MONOIDAL AND CLOSED POSETS

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C.1

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by

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A THESIS

SUBMITTED TO THE COMMITTEE ON GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

ST. JOHN'S, NEWFOUNDLAND

DECEMBER, 1974

This thesis has been examined and approved by:

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ACKNOWLEDGEMENTS

I am sincerely grateful to Dr. Peter Booth, my Supervisor, for his continual guidance and help in the writing of this thesis. I would also like to thank Mrs. Hilda Tiller for typing the manuscript.

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ABSTRACT

Any poset can be regarded as a category, hence the concepts of monoidal, closed, and monoidal closed categories give rise to corresponding concepts of structured posets. The thesis is a study of these enriched posets, it includes a detailed account of relations between the cardinality of the sets of structured posets that can be carried by a given poset.

INTRODUCTION

It is well known that partially ordered sets (posets) can be regarded as categories, treating their elements as objects and the valid $\leq$'s as morphisms. Eilenberg and Kelly [5] introduced the concept of monoidal, symmetric monoidal, closed, monoidal closed, and symmetric monoidal closed categories; if the categories used are posets then we have the associated concepts of monoidal, symmetric monoidal, closed, monoidal closed, and symmetric monoidal closed posets. If $(P,\leq)$ is a poset then $M(P,\leq)$, $SM(P,\leq)$, $C(P,\leq)$, $SMC'(P,\leq)$ will denote the sets of monoidal, symmetric monoidal, closed and symmetric monoidal non-closed structures on $(P,\leq)$ respectively, etc. The examples section of [5] contains a brief account of monoidal and monoidal closed posets; in this thesis we develop certain aspects of their theory in more detail.

In Chapter 1 the basic language of posets is given together with the definitions of monoidal poset $(P,\leq,\otimes,k)$ and symmetric monoidal poset. A key result (Theorem 1.2.1) asserts that if a monoidal poset has first element 0 and last element ω then $0 \otimes \omega = 0$, $0 \otimes \omega = \omega$ or $0 \otimes \omega$ is incomparable with the identity element k, and a similar result applies to $\omega \otimes 0$.

Any bimorphism $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$ has an associated opposite bimorphism $\otimes_o$: if $(P,\leq)$ has an inversion μ then two more bimorphisms associated with $\otimes$, i.e. $\bar{\otimes}$ and $\otimes_\mu$, are defined. The functions $\otimes \to \otimes_o$, $\otimes \to \bar{\otimes}$, and $\otimes \to \otimes_\mu$ allow us to relate the cardinality of the sets $M(P,\leq)$ and $SM(P,\leq)$ and their subsets of monoidal posets with $0 \otimes \omega = 0$. The reader will notice that in this thesis the function $\otimes \to \bar{\otimes}$ is the most useful of these relationships.

The results up to §1.4 Proposition 1.4.3 (apart from §1.3 and certain parts of §1.1) are essential preliminaries to Chapters III and IV of this thesis. The remainder of §1.4, §1.5, §1.6 is devoted to the study of $M(P,\leq)$ and is not essential for an understanding of later chapters (with the single exception of Theorem 1.6.6 which is used in the proof of Corollary 4.4.5). Results from these last parts of Chapter I include the fact that if $(P,\leq)$ is a poset satisfying suitable conditions then $\#(M(P,\leq))$ and $\#(SM(P,\leq))$ are even, and under stronger conditions $\{\#(M(P,\leq)) - \#(SM(P,\leq))\}$ is divisible by 4.

In Chapter II the definition of left and right adjoints of morphisms and bimorphisms of posets are given. Theorems are then proved determining when morphisms and bimorphisms have left adjoints and right adjoints. In particular it is shown that if $(P,\leq)$ has a first element 0 and a last element ω and the bimorphisms $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$ has a right adjoint then $0 \otimes \omega = 0$; also that the converse holds if $(P,\geq)$ is well ordered (Theorem 2.2.6). A similar result is proved for bimorphisms $\pitchfork : (P,\leq) \times (P,\leq) \to (P,\leq)$.

In Chapter III the definitions of closed poset $(P,\leq,\pitchfork,k)$ and the related concepts of monoidal closed, symmetric monoidal closed, cartesian closed and biclosed posets are given. One of the main results here is Theorem 3.1.1, part (f) which states that if $(P,\leq,\pitchfork,k)$ is a closed poset with 0 and ω then $\omega \pitchfork \omega = \omega$ or k is incomparable with $\omega \pitchfork (\omega \pitchfork \omega)$. A corollary to this result (Corollary 3.1.3) asserts that if $(P,\leq)$ is well ordered then $\pitchfork$ has a left adjoint bimorphism $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$. The data of a monoidal closed poset is partially redundant, so the relations between the data are studied. Theorem 3.3.1 is a particular case of a standard result about monoidal categories; it asserts that if $(P, \leq, \otimes, k)$ is a monoidal poset and $\otimes$ has a right adjoint then $(P, \leq, \otimes, k)$ is expandable into a monoidal closed poset. This links up with Theorem 1.2.1 and Theorem 3.2.6 and allows us to relate the cardinality of certain sets of structures on posets in 3.4. The main result in 3.4 is Theorem 3.4.2 which states that if $(P, \leq)$ is a non-trivial chain with inversion u, first element 0 (and hence a last element ∞) then the rule $(\otimes \mapsto \bar{\otimes})$ determines bijections (i) $MC(P,\leq) \to MC'(P,\leq)$, (ii) $SMC(P,\leq) \to SMC'(P,\leq)$, (iii) $S'MC(P,\leq) \to S'MC'(P,\leq)$. Several corollaries follow from this result showing that :-

(i) $\#(M(P,\leq)) = 2 \times \#(MC(P,\leq)) = 2 \times \#(MC'(P,\leq))$ (Corollary 3.4.3) ;

(ii) $\#(SM(P,\leq)) = 2 \times \#(SMC(P,\leq)) = 2 \times \#(SMC'(P,\leq))$ (Corollary 3.4.3) ;

(iii) $\#(M(P,\leq)) \leq 2 \times \#(C(P,\leq))$ (Corollary 3.4.4) ;

and (iv) under suitable conditions $2 \mid \#(S'MC(P,\leq))$ and $2^n \mid \#(S'MC'(P,\leq))$ (Corollary 3.4.5).

Chapter IV gives a procedure for obtaining all monoidal and closed structures on a finite chain, based on Theorem 3.1.2. Lists of all monoidal, monoidal closed, closed, etc. structures carried by the ordinal number posets 2, 3, 4 are given. The numbers involved illustrate our previous results. Also examples of two interesting structures are given, namely,

(a) closed posets which have an associated left adjoint $\otimes$ but are not monoidal closed because $\otimes$ is non-associative,

(b) non-symmetric monoidal posets which are biclosed posets.

Chapter V is mainly taken from the paper "Closed Categories" by Eilenberg and Kelly [3] and gives the basic definition of monoidal, symmetric monoidal, closed, monoidal closed, symmetric monoidal closed categories and related material. Then in §5.5 we analyse the relationships existing between poset structures and the corresponding category structures; it is obvious that a monoidal structure on a poset is our "monoidal poset", the main result here is (Theorem 5.5.6) that a closed structure on a poset is our "closed poset". We conclude Chapter V with a remark concerning the apparent "symmetry" between the definitions of monoidal and closed categories.

Remark: There are two obvious ways of generalizing the concept of poset which might seem appropriate here.

(i) If we work in a von Neumann-Bernays-Gödel type set theory we could assume that P was a class rather than a set. This leads to logical difficulties as soon as we try to refer to $M(P,\leq)$, $MC(P,\leq)$, etc., and has therefore been avoided. If, on the other hand, we assume Zermelo-Fraenkel axioms and the existence of a universe as in [7] then this question does not arise.

(ii) We could have assumed that $\leq$ was only a preorder, i.e. dropped condition (iii) in the definition of poset. This leads to essentially the same theory, for a given monoidal preorder $(P,\leq,\otimes,k)$, we can factor out by the equivalence relation $\sim$ on P, $a \sim b$, if $a \leq b$ and $b \geq a$ and obtain a monoidal poset $(P/(\sim), \leq', \otimes', [k])$ [3, pp. 554-5].

CHAPTER I

MONOIDAL POSETS

1.1 Basic Definitions

Definition 1.1.1 A partially ordered set (poset) $(P,\leq)$ is a set P with a binary relation $x \leq y$ such that for all $x,y,z \in P$,

(i) $x \leq x$;

(ii) if $x \leq y$ and $y \leq z$ then $x \leq z$;

(iii) if $x \leq y$ and $y \leq x$ then $x = y$.

Definition 1.1.2 A trivial poset is a poset with exactly one element.

Definition 1.1.3 A chain is a poset $(P,\leq)$ which satisfies

(iv) for all $x,y \in P$, either $x \leq y$ or $y \leq x$.

Definition 1.1.4 Let X be a subset of a poset $(P,\leq)$. Then an element $a \in X$ is a first element of X if $a \leq x$ for all $x \in X$. An element $b \in X$ is a last element of X if $b \geq x$ for all $x \in X$.

Definition 1.1.5 A poset $(P,\leq)$ is well ordered if every non-void subset of $(P,\leq)$ has a first element.

It is easily seen that any well ordered poset is a chain.

Definition 1.1.6 The dual of a poset $(P,\leq)$ is the poset $(P,\geq)$ defined by the converse partial ordering relation on the same set.

Posets are dual in pairs. Definitions and theorems about posets are dual in pairs; and if any theorem is true for all posets so is its dual. For a dual proof we take the same statement and proof as before except that certain specified terms are changed, i.e.

$\leq$ becomes $\geq$; $\geq$ becomes $\leq$; $<$ becomes $>$; $>$ becomes $<$; first element becomes last element and so on.

Definition 1.1.7 The <u>product</u> $(P,\leq) \times (P',\leq)$ of two posets $(P,\leq)$ and $(P',\leq)$ is the set of all pairs (x,y) with $x \in P$ and $y \in P'$, such that $(x_1,y_1) \leq (x_2,y_2)$ if and only if $x_1 \leq x_2$ in $(P,\leq)$ and $y_1 \leq y_2$ in $(P',\leq)$.

Definition 1.1.8 Let x and y be two elements of a poset $(P,\leq)$. An element $b \in P$ is a <u>lower bound</u> for x and y in P when $b \leq x$ and $b \leq y$. An element $m \in P$ is a <u>meet</u> of x and y, denoted $m = x \wedge y$, when it is a lower bound of x and y and $m \geq b$ for all lower bounds b of x and y. It is easily seen that the meet of two elements is unique.

Definition 1.1.9 <u>A poset $(P,\leq)$ will be said to have a meet $\wedge$</u> if $a \wedge b$ exists for all $a,b \in P$. Such a $(P,\leq,\wedge)$ will be called a <u>meet-semilattice</u> [5, p.8]. We notice that $\wedge$ is then a well defined binary operation, $\wedge : P \times P \to P$.

Definition 1.1.10 Let x and y be two elements of a poset $(P,\leq)$. An element $u \in P$ is an <u>upper bound</u> for x and y when both $x \leq u$ and $y \leq u$. An element $\ell \in P$ is a <u>join</u> of x and y, denoted $\ell = x \vee y$ when it is an upper bound and $\ell \leq u$ for all other upper bounds of x and y. It is easily seen that the join of two elements is unique.

Definition 1.1.11 <u>A poset $(P,\leq)$ will be said to have a join $\vee$</u> if $a \vee b$ exists for all $a,b \in P$. Such a $(P,\leq,\vee)$ will be called a <u>join-semilattice</u> [5, p.8]. We notice that $\vee$ is then a well defined binary operation, $\vee : P \times P \to P$.

Definition 1.1.12 A <u>lattice</u> is a poset $(P, \leq)$ with both a meet and a join. From Definitions 1.1.9 and 1.1.11 it follows that a poset $(P, \leq)$ is a lattice if and only if it is a meet- and a join- semilattice.

Definition 1.1.13 Let $(P, \leq)$ and $(P', \leq)$ be posets. A <u>morphism of posets</u> is a function $\theta : P \to P'$ such that for all $x, y \in P$:

$$x \leq y \Rightarrow \theta(x) \leq \theta(y).$$

Definition 1.1.14 A morphism of posets $\theta : P \to P'$ is an <u>isomorphism</u> if θ is a bijection and

$$x \leq y \Leftrightarrow \theta(x) \leq \theta(y).$$

Definition 1.1.15 Two posets $(P, \leq)$ and $(P', \leq)$ are <u>isomorphic</u> if there exists an isomorphism of posets $\theta : P \to P'$.

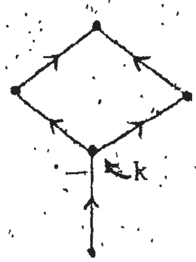
Definition 1.1.16 Given posets $(P, \leq)$, $(Q, \leq)$, $(R, \leq)$, a <u>bimorphism</u> $\otimes : (P, \leq) \times (Q, \leq) \to (R, \leq)$ is a morphism : $(P \times Q, \leq) \to (R, \leq)$, i.e. a binary operation $\otimes : P \times Q \to R$ such that if $a \leq b$ in $(P, \leq)$ and $a' \leq b'$ in $(Q, \leq)$ then $a \otimes a' \leq b \otimes b'$ in $(R, \leq)$. This is equivalent to assuming $a \leq b \Rightarrow a \otimes a' \leq b \otimes a'$ for all $a, b \in P$, $a' \in Q$ and $a' \leq b' \Rightarrow a \otimes a' \leq a \otimes b'$ for all $a \in P$, $a', b' \in Q$.

Definition 1.1.17 If $(P, \leq)$ is a poset and $x, y \in P$ then x is said to be <u>comparable</u> with y if either $x \leq y$ or $x \geq y$, otherwise they are said to be <u>incomparable</u>.

Definition 1.1.18 If $(P, \leq)$ is a poset and $k \in P$, then $(P, \leq)$ is said to be <u>k-comparable</u> if every element of P is comparable with k.

Examples: (i) If $(P,\leq)$ is a chain and $k \in P$, then $(P,\leq)$ is k-comparable.

(ii) The following poset is k-comparable:



(With poset diagrams of this type we follow the obvious convention that the points $\bullet$ of the diagram are poset elements, and "paths in the directions of arrows" are valid $\leq$).

Definition 1.1.19. An <u>inversion</u> for the poset $(P,\leq)$ is a function $\mu : P \to P$ such that

(i) $\mu : (P,\leq) \to (P,\geq)$ is a morphism

and (ii) $\mu \circ \mu = 1_P$.

Examples:

(a) The ordinal number n is the chain $\{0, 1, 2, \ldots, n-1\}$. If $(P,\leq)$ is the ordinal number $\{0, 1, 2, \ldots, n-1\}$ then the function $\mu : P \to P$, $\mu(x) = n - 1 - x$, $x \in P$ is the unique associated inversion.

(b) Any finite chain is, to within isomorphism, a finite ordinal number; so, finite chains have inversions.

(c) Let $(L,\leq,\wedge)$ be a meet-semilattice (see Definition 1.1.9) with first element 0. An element a^* is a pseudocomplement of a $(\in L)$ if (i) $a \wedge a^* = 0$ and (ii) $a \wedge x = 0$ implies that $x \leq a^*$. <u>A pseudocomplemented-meet-semilattice</u> is one in which every element has a pseudocomplement. Any pseudocomplemented meet-semilattice has an inversion, namely

$\mu(a) = a^*$, $a \in L$. [5, pp.58-9].

In particular, any Boolean Algebra B [5, p. 58] has an inversion μ where $\mu(x) = x'$ the complement of x, $x \in B$.

(d) An ordered group $(X, \leq, \cdot)$ consists of two structures, a poset $(X, \leq)$ and a group $(X, \cdot)$, related by

(i) $a \leq b \Rightarrow a \cdot c \leq b \cdot c$,

and $c \cdot a \leq c \cdot b$,

and (ii) $a \leq b \Rightarrow a^{-1} \geq b^{-1}$, for all $a, b, c \in P$;

(compare with [3, p. 555]). Given an element m in the centre of X (such that $m \cdot x = x \cdot m$ for all $x \in X$), we define $\mu_m : X \to X$ by $\mu_m(x) = m \cdot x^{-1}$. μ_m is an inversion since

$$
\begin{aligned}
\text{(i) } \mu_m \circ \mu_m(x) &= m \cdot (m \cdot x^{-1})^{-1} \\
&= m \cdot (x \cdot m^{-1}) \\
&= (m \cdot x) \cdot m^{-1} \\
&= (x \cdot m) \cdot m^{-1} \\
&= x \cdot (m \cdot m^{-1}) \\
&= x.
\end{aligned}
$$

and (ii) $x \leq y \Rightarrow x^{-1} \geq y^{-1}$

$m \cdot x^{-1} \geq m \cdot y^{-1}$

$\mu_m(x) \geq \mu_m(y)$ (see Definition 1.1.19).

Examples of ordered groups include:

(i) $X = A^n$ where A is the set of integers or the set of rationals or the set of reals, n is a natural number and the binary operation is addition.

(ii) Any group where $a \leq b$ is defined to mean a and b are identical.

Lemma 1.1.1 Let the poset $(P,\leq)$ have first element 0 and last element ω. Then if $(P,\leq)$ has an inversion μ

(a) $\mu 0 = \omega$.

(b) $\mu\omega = 0$.

Proof: (a) $0 \leq \mu\omega$ since 0 is the first element of $(P,\leq)$.

$\mu 0 \geq \mu(\mu\omega)$

$\mu 0 \geq \omega$

$\mu 0 = \omega$ since ω is the last element of $(P,\leq)$.

(b) $\mu\omega = \mu\mu(0) = 0$.

Definition 1.1.20 Let X be a finite set and $f : X \to X$ a function, then $x \in X$ is said to be a <u>fixed point</u> of f if $f(x) = x$.

Example: Let $(A,\leq,\cdot)$ be an ordered group, m be an element of the centre of A, and $\mu_m : A \to A$ be the associated inversion. Then $a \in A$ is a fixed point for $\mu_m \iff \mu_m(a) = a \iff m \cdot a^{-1} = a \iff m = a^2$. Hence μ_m has a fixed point if and only if m has a square root.

In particular for the ordered group $\mathbb{Z}^n$ of n-tuples of integers with addition, μ_m is an inversion for all $m \in \mathbb{Z}^n$ and μ_m has a fixed point $\iff m = (m_1, m_2, \ldots, m_n)$, and each m_i is even.

Notation $\#(X)$ will denote the cardinal number of the set X.

Proposition 1.1.2 If X is a finite set and $f : X \to X$ is a function such that $f \circ f = 1_X$ and f has no fixed points then $\#(X)$ is even.

Proof: f associates the elements of X in pairs of distinct elements and the result follows.

Definition 1.1.21 The poset $(P, \leq)$ is said to be fixed point comparable if it has an inversion μ with the property that any existing fixed point for μ is comparable with every other element of P.

Proposition 1.1.3 If $(P, \leq)$ is a poset which is fixed point comparable with respect to μ, then μ has either one fixed point or none at all.

Proof: Suppose k_1 and k_2 are two fixed points with respect to μ. Then $k_1 \leq k_2 \Rightarrow k_1 = \mu(k_1) \geq \mu(k_2) = k_2$, and so $k_1 = k_2$.

Proposition 1.1.4 If the finite poset $(P, \leq)$ with inversion μ is fixed point comparable, then

$$\#(P) \text{ is even} \iff \mu \text{ has no fixed points,}$$

$$\#(P) \text{ is odd} \iff \mu \text{ has one fixed point.}$$

Proof: f. associates the non-fixed points in pairs and the result follows from Proposition 1.1.3.

Examples:

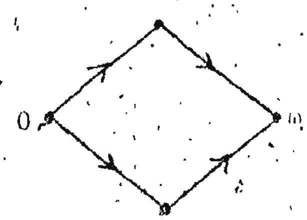
1. All chains with inversions are fixed point comparable.

We notice that:

(a) Finite chains with an even number of elements have no fixed points.

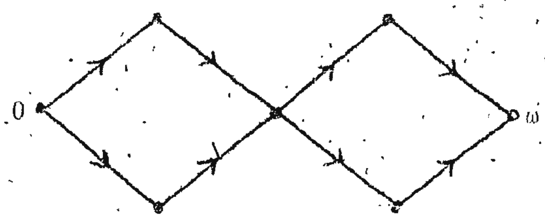
(b) Finite chains with an odd number of elements have one fixed point - the "middle element" - which of course is comparable.

2.



This poset, which is not a chain, has two inversions. One inversion has two incomparable fixed points; the other has no fixed points, hence the poset is fixed point comparable.

3.



The reader can easily see that the poset illustrated above has two associated inversions. In each case there is one fixed point - the middle element - and this is comparable with all other elements. Hence the poset is fixed point comparable.

§1.2 Monoidal Posets, First and Last Elements

Definition 1.2.1 A <u>monoidal poset</u> $(P, \leq, \otimes, k)$ [3,p.554] consists of

(i) a poset $(P, \leq)$;

(ii) an associative bimorphism $\otimes : (P, \leq) \times (P, \leq) \rightarrow (P, \leq)$

i.e.

$$(a \otimes b) \otimes c = a \otimes (b \otimes c) \quad \text{for all } a,b,c \in P;$$

(iii) an element $k \in P$ such that

$$a \otimes k = a \quad \text{for all } a \in P ;$$

$$k \otimes a = a \quad \text{for all } a \in P .$$

Notation $M(P,\leq)$ will denote the set of all monoidal structures on the poset $(P,\leq)$.

Definition 1.2.2 A monoidal poset $(P,\leq,\otimes,k)$ in which $a \otimes b = b \otimes a$ for all $a,b \in P$ is called a <u>symmetric monoidal poset</u>.

Notation $SM(P,\leq)$ will denote the set of all symmetric monoidal structures on the poset $(P,\leq)$.

Examples:

1. Let $(X,\leq,\cdot)$ be an ordered group (see page 5, example (d)). If we take $a \otimes b$ to be ab and k to be the identity, then we have a monoidal structure which is symmetric if the group is abelian. [3, p. 555].

2. If $(P,\leq,\wedge)$ is a meet-semilattice (see Definition 1.1.9) with last element ω then $(P,\leq,\wedge,\omega)$ is a symmetric monoidal structure.

3. If $(P,\leq,\vee)$ is a join semilattice (see Definition 1.1.11) with first element 0, then $(P,\leq,\vee,0)$ is a symmetric monoidal structure.

Theorem 1.2.1 If $(P,\leq,\otimes,k)$ is a monoidal poset with first element 0 and last element ω then

(a) $0 \otimes 0 = 0$;

(b) $\omega \otimes \omega = \omega$;

(c) either $0 \otimes \omega = 0$ or $0 \otimes \omega = \omega$ or $0 \otimes \omega$ is incomparable with k;

(d) either $\omega \otimes 0 = 0$ or $\omega \otimes 0 = \omega$ or $\omega \otimes 0$ is incomparable with k.

Proof: (a) $0 \otimes 0 \leq 0 \otimes k = 0$ therefore $0 \otimes 0 = 0$.

(b) $\omega \otimes \omega \geq \omega \otimes k = \omega$ therefore $\omega \otimes \omega = \omega$.

(c) Assume $0 \otimes \omega$ is k-comparable, then $0 \otimes \omega \leq k$ or $0 \otimes \omega \geq k$.

Case 1. Assume $0 \otimes \omega \leq k$.

Then by part (a) and by definition of an associative $\otimes$

$0 \otimes \omega = (0 \otimes 0) \otimes \omega = 0 \otimes (0 \otimes \omega) \leq 0 \otimes k = 0$.

Therefore $0 \otimes \omega = 0$ since 0 is the first element of P.

Case 2. Assume $0 \otimes \omega \geq k$.

Then by part (b) and by the definition of an associative $\otimes$

$0 \otimes \omega = 0 \otimes (\omega \otimes \omega) = (0 \otimes \omega) \otimes \omega \geq k \otimes \omega = \omega$.

Therefore $0 \otimes \omega = \omega$ since ω is the last element of P.

(d) Similarly (d) can be proved by applying part (c) to $(P,\geq)$ i.e., it is the dual result.

Corollary 1.2.2 If $(P,\leq)$ is k-comparable (see page 3) then

(a) either $0 \otimes \omega = 0$ or $0 \otimes \omega = \omega$.

(b) either $\omega \otimes 0 = 0$ or $\omega \otimes 0 = \omega$.

Proof: This follows immediately from Theorem 1.2.1.

§1.3 The Bimorphism $\otimes_o$ and $\{\#(M(P,\leq)) - \#(SM(P,\leq))\}$

Let $(P,\leq)$ be a poset. Let $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$ be a bimorphism and define $\otimes_o : (P,\leq) \times (P,\leq) \to (P,\leq)$ be the "opposite bimorphism" such that $a \otimes_o b = b \otimes a$. Then with this basic situation we have the following proposition:

Proposition 1.3.1 (i) If $(P,\leq,\otimes,k)$ is a monoidal poset then so is $(P,\leq,\otimes_o,k)$.

(ii) If $(P,\leq,\otimes,k)$ is symmetric monoidal, these monoidal posets are identical, otherwise they are distinct.

Proof: This is immediate from the definitions involved.

Notation We use $S'M(P,\leq)$ to denote the set of all monoidal structures that are not symmetric on the poset $(P,\leq)$.

Theorem 1.3.2 If $(P,\leq)$ is a poset such that the set of non-symmetric monoidal structures on $(P,\leq)$ is finite (i.e. $S'M(P,\leq)$ is finite) then $\#(S'M(P,\leq))$ is even, i.e.

$$2 \mid \#(S'M(P,\leq)) \text{ or equivalently}$$
$$2 \mid \{\#(M(P,\leq)) - \#(SM(P,\leq))\}.$$

Proof: The result follows from Proposition 1.1.2, where $X = S'M(P,\leq)$ and f is the function: $\otimes \mapsto \otimes_0$, for by Proposition 1.3.1 each linked pair of monoidal structures is distinct.

Remarks: (i) The rather obvious results in this section will be useful in §1.6 and in results concerning biclosed posets.

(ii) If $(P,\leq)$ is a finite poset, then it is obvious that $S'M(P,\leq)$ is finite and hence the result of Theorem 1.3.2 will follow.

<u>For the remainder of Chapter I we assume that $(P,\leq)$, the poset used, will have a given inversion μ.</u>

§1.4 The Bimorphism $\bar{\otimes}$, $\#(M(P,\leq))$ and $\#(SM(P,\leq))$

Let $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$ be a bimorphism and define $\bar{\otimes} : (P,\leq) \times (P,\leq) \to (P,\leq)$ to be the bimorphism such that $a \;\bar{\otimes}\; b = \mu(\mu b \otimes \mu a)$.

Lemma 1.4.1 $\bar{\bar{\otimes}} = \otimes$.

Proof: $a \,\bar{\bar{\otimes}}\, b = \mu(\mu b \,\bar{\otimes}\, \mu a) = \mu\mu(\mu\mu a \otimes \mu\mu b) = a \otimes b$.

Lemma 1.4.2 Let $(P, \leq, \otimes, k)$ be a monoidal poset with first element 0 and last element ω. Then

(a) If $0 \otimes \omega = 0$ then $0 \,\bar{\otimes}\, \omega = \omega$.

(b) If $0 \otimes \omega = \omega$ then $0 \,\bar{\otimes}\, \omega = 0$.

Proof: (a) If $0 \otimes \omega = 0$ then
$$\begin{aligned} 0 \,\bar{\otimes}\, \omega &= \mu(\mu\omega \otimes \mu 0) \\ &= \mu(0 \otimes \omega) \\ &= \mu 0 \\ &= \omega . \end{aligned}$$

(b) If $0 \otimes \omega = \omega$ then
$$\begin{aligned} 0 \,\bar{\otimes}\, \omega &= \mu(\mu\omega \otimes \mu 0) \\ &= \mu(0 \otimes \omega) \\ &= 0. \end{aligned}$$

Proposition 1.4.3 If $(P, \leq, \otimes, k)$ is a monoidal poset, then $(P, \leq, \bar{\otimes}, \mu k)$ is also a monoidal poset. If $(P, \leq, \otimes, k)$ is a symmetric monoidal poset, then so is $(P, \leq, \bar{\otimes}, \mu k)$.

Proof: This follows immediately from the definitions involved.

Remark: The results up to this point (apart from §1.3 and some parts of §1.1) are essential preliminaries to the results of Chapter III and Chapter IV ; the remainder of Chapter I is devoted to a further study of $M(P,\leq)$ but is not an essential preliminary to results of later chapters (with the single exception of Theorem 1.6.6 which is required in the proof of Corollary 3.4.5).

Proposition 1.4.4 Given $(P, \leq, \otimes, k)$ is a monoidal poset then $\otimes = \bar{\otimes}$ implies $k = \mu k$.

Proof: $k = k \mathbin{\bar{\otimes}} \mu k = k \otimes \mu k = \mu k$.

Proposition 1.4.5 Given $(P, \leq, \otimes, k)$ is a non-trivial, monoidal poset with first element 0 and last element ω then $\otimes = \bar{\otimes}$ implies $0 \otimes \omega$ and $\omega \otimes 0$ are incomparable with k.

Proof: By Lemma 1.4.2 if $0 \otimes \omega = 0$ then $0 \mathbin{\bar{\otimes}} \omega = \omega$; if $0 \otimes \omega = \omega$ then $0 \mathbin{\bar{\otimes}} \omega = 0$. By Theorem 1.2.1 $0 \otimes \omega = 0$ or $0 \otimes \omega = \omega$ or $0 \otimes \omega$ is incomparable with k. If $\otimes = \bar{\otimes}$ a contradiction results unless $0 \otimes \omega$ is incomparable with k. A similar argument shows $\omega \otimes 0$ is incomparable with k.

Proposition 1.4.6 Let $(P, \leq, \otimes, k)$ be a monoidal poset.

If either (i) k is not a fixed point for μ;

or (ii) $(P, \leq)$ is non-trivial, k-comparable, with first element 0 and last element ω;

then $\otimes \neq \bar{\otimes}$.

Proof: (i) and (ii) follow from Proposition 1.4.4 and Proposition 1.4.5 respectively.

Theorem 1.4.7 Let $(P, \leq)$ be a non-trivial poset such that $M(P, \leq)$ is finite. If either (i) μ has no fixed points (if P is a finite poset this is equivalent to assuming $\#(P)$ is even, see Proposition 1.1.4);

or (ii) $(P, \leq)$ is a chain with 0 and ω;

then $\#(M(P, \leq))$ is even.

Proof: The result follows from Proposition 1.1.2 where $X \neq M(P,\leq)$ and f is the function: $\otimes \to \bar{\otimes}$, and from Proposition 1.4.6 which shows that each pair consists of two distinct posets.

Theorem 1.4.8 Let $(P,\leq)$ be a non-trivial poset such that $SM(P,\leq)$ is finite. If either (i) μ has no fixed points ;

or (ii) $(P,\leq)$ is a chain with 0 and ω;

then $\#(SM(P,\leq))$ is even.

Proof: This follows by the argument used to prove Theorem 1.4.7, where $M(P,\leq)$ is replaced by $SM(P,\leq)$, and the fact that $\otimes$ is commutative if and only if $\bar{\otimes}$ is commutative.

§1.5 The Bimorphism $\otimes_\mu$ and $\#(HM(P,\leq))$

Let $(P,\leq)$ be a poset. If $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$ is a bimorphism we define $\otimes_\mu : (P,\leq) \times (P,\leq) \to (P,\leq)$ to be the bimorphism such that $a \otimes_\mu b = \mu(\mu a \otimes \mu b)$.

Definition 1.5.1 If $(P,\leq)$ is a poset and $\otimes$ is a bimorphism on $(P,\leq)$ then $\otimes$ will be said to be "homomorphic" with respect to the inversion μ if $\mu : (P,\leq) \to (P,\geq)$ is a homomorphism i.e. $\mu(a \otimes b) = \mu a \otimes \mu b$ for all $a,b \in P$.

Proposition 1.5.1 (i) If $(P,\leq,\otimes,k)$ is a monoidal poset then $(P,\leq,\otimes_\mu,\mu k)$ is a monoidal poset.

(ii) If $\otimes$ is homomorphic then these two monoidal structures are identical, otherwise they are distinct.

Proof: (i) This follows immediately from the definitions involved.

(ii) $a \otimes b = a \otimes_\mu b$ (i.e. $a \otimes b = \mu(\mu a \otimes \mu b)$) if and only if $\otimes$ is homomorphic, and the result follows because monoids have unique two-sided identities.

Corollary 1.5.2 (i) $\mu(k) \neq k \Rightarrow \otimes$ is not homomorphic.

(ii) If μ has no fixed points then $(P,\leq)$ carries no homomorphic monoidal structures.

Proof: (i) If $\otimes$ is homomorphic then k is a fixed point for μ (see Proposition 1.5.1 (ii)), and the result follows.

(ii) is then immediate.

Remark: This result (Corollary 1.5.2 part (ii)) applies to

(i) finite posets $(P,\leq)$ where $\#(P)$ is even (Proposition 1.1.4);

(ii) ordered groups with inversion μ_m where m has no square roots (see example on page 6).

Notation $HM(P,\leq)$ will denote the set of all homomorphic monoidal structures on the poset $(P,\leq)$.

$H'M(P,\leq)$ will denote the set of all non-homomorphic monoidal structures on the poset $(P,\leq)$.

Proposition 1.5.3: If $(P,\leq)$ is a poset such that the set $(H'M(P,\leq))$ is finite then $\#(H'M(P,\leq))$ is even i.e.

$$2 \mid \#(H'M(P,\leq)), \text{ or equivalently}$$

$$2 \mid \{\#(M(P,\leq)) - \#(HM(P,\leq))\}.$$

Proof: The result follows from Proposition 1.1.2 where $X = H'M(P,\leq)$ and f is the function: $\otimes \mapsto \otimes_\mu$. It follows from Proposition 1.5.1 that each pair of monoidal structures thus linked is distinct.

Corollary 1.5.4 Let $(P,\leq)$ be a non-trivial poset such that $M(P,\leq)$ is finite. If either (i) μ has no fixed points;

or (ii) $(P,\leq)$ is a chain with 0 and ω;

then $\#(HM(P,\leq))$ is even.

Proof: This follows from Theorem 1.4.7 and Proposition 1.5.3.

Remark: The result of Theorem 1.4.7 (i) can be obtained using $\otimes_\mu$ instead of $\bar{\otimes}$. If μ has no fixed points then by Corollary 1.5.2 part (ii), $(P,\leq)$ carries no homomorphic monoidal structures, i.e. $\#(HM(P,\leq)) = 0$. By Proposition 1.5.3 $\#(M(P,\leq))$ is even.

§1.6 The Bimorphisms $\otimes$, $\otimes_o$, $\bar{\otimes}$, $\otimes_\mu$ and $\{\#(M(P,\leq)) - \#(SM(P,\leq))\}$.

We first relate the four bimorphisms $\otimes$, $\otimes_o$, $\bar{\otimes}$, $\otimes_\mu$. Given a monoidal structure $(P,\leq,\otimes,k)$ we can construct associated bimorphisms $\otimes_o$, $\otimes_\mu$, $(\otimes_o)_\mu$, $(\otimes_\mu)_o$, $((\otimes_o)_\mu)_o$, etc. which are best abbreviated to $\otimes_o$, $\otimes_\mu$, $\otimes_{o\mu}$, $\otimes_{\mu o}$, $\otimes_{o\mu o}$, etc.

Lemma 1.6.1 (a) $\otimes_{o\mu} = \otimes_{\mu o} = \bar{\otimes}$.

(b) $\otimes_{oo} = \otimes_{\mu\mu} = \otimes$.

Proof: (a) $a \otimes_{o\mu} b = \mu(\mu a \otimes_o \mu b) = \mu(\mu b \otimes \mu a) = a \bar{\otimes} b$.

$a \otimes_{\mu o} b = b \otimes_\mu a = \mu(\mu b \otimes \mu a) = a \bar{\otimes} b$.

(b) Obvious.

Proposition 1.6.2 If $(P,\leq)$ is a poset and $(P,\leq,\otimes,k)$ is a monoidal poset then the set of four monoidal posets $\{(P,\leq,\otimes,k)$, $(P,\leq,\otimes_o,k)$,

$(P,\leq,\overline{\otimes}, \mu k)$; $(P,\leq,\otimes_\mu, \mu k)\}$ is a "closed system" in the sense that if we use the three rules:

(i) $\otimes \to \otimes_o$, $k \to k$;

(ii) $\otimes \to \overline{\otimes}$, $k \to \mu k$;

(iii) $\otimes \to \otimes_\mu$, $k \to \mu k$;

to generate new monoidal posets then no matter which of the four monoidal structures we start with, we always end up with one of the original four monoidal posets.

Proof: This follows from Lemma 1.6.1 by inspection of all cases.

Proposition 1.6.3 Let $(P,\leq,\otimes,k)$ be a monoidal poset, such that $(P,\leq)$ is non-trivial, k-comparable with first element 0 and last element ω then

(i) $\otimes$ can not both be commutative and homomorphic.

(ii) $\otimes$ is commutative if and only if $\otimes = \otimes_o \neq \overline{\otimes} = \otimes_\mu$.

(iii) $\otimes$ is homomorphic if and only if $\otimes = \otimes_\mu \neq \otimes_o = \overline{\otimes}$.

(iv) $\otimes$ is neither commutative nor homomorphic if and only if $\otimes$, $\otimes_o$, $\overline{\otimes}$ and $\otimes_\mu$ are all distinct.

Proof: (i) If $\otimes$ is both commutative and homomorphic then $a \otimes b = \mu\mu a \otimes \mu\mu b = \mu(\mu a \otimes \mu b) = \mu(\mu b \otimes \mu a) = a \overline{\otimes} b$, which contradicts Proposition 1.4.6 (ii).

(ii), (iii), and (iv) follow readily from the definitions involved and part (i).

Proposition 1.6.4 Let $(P,\leq,\otimes,k)$ be a monoidal poset such that k is not a fixed point for μ. Then $\otimes$ satisfies either condition (ii) or condition (iv) of Proposition 1.6.3.

Proof: This follows from Corollary 1.5.2 (i) and Proposition 1.4.6 (i).

Examples:

If $\otimes$ is the meet, $\wedge$, in the monoidal poset $(P, \leq, \wedge, \omega)$ then $\otimes = \otimes_o = \wedge$, and $\bar{\otimes} = \otimes_\mu = \vee$.

If $\otimes$ is the join, $\vee$, in the monoidal poset $(P, \leq, \vee, 0)$ then $\otimes = \otimes_o = \vee$, and $\bar{\otimes} = \otimes_\mu = \wedge$.

Remark: Suppose we have an ordered group $(X, \leq, \cdot)$ with inversion $\mu_m(x) = m \cdot x^{-1}$ (see example (d) page 5); then there is an associated monoidal poset with $\otimes = \cdot$ (see example 1 page 9).

(i) $\otimes = \otimes_o$ if and only if $\cdot$ is commutative.

(ii) $\otimes = \bar{\otimes}$ if and only if m = the identity for $(X, \cdot)$.

(iii) $\otimes = \otimes_o = \bar{\otimes} = \otimes_\mu$ if and only if $\cdot$ is commutative and m is the identity for $(X, \cdot)$.

Hence the relations between $\otimes$, $\otimes_o$, $\bar{\otimes}$ and $\otimes_\mu$ for ordered groups are quite different from those described in Proposition 1.6.3 and Proposition 1.6.4. It follows that the only ordered group structure on a finite chain is the trivial ordered group (on the chain of one element); though this can, of course, easily be seen to follow from the definitions without introducing our approach.

Notation. We use $H'S'M(P, \leq)$ to denote the set of all non-homomorphic, non-symmetric monoidal structures on the poset $(P, \leq)$.

Proposition 1.6.5 Let $(P, \leq)$ be a poset such that $H'S'M(P, \leq)$ is finite. If either (i) μ has no fixed points

or (ii) $(P, \leq)$ is a non-trivial chain with first element 0 and last element ω

then $4 \mid \#(H'S'M(P,\leq))$.

Proof: Proposition 1.6.2 implies that $H'S'M(P,\leq)$ can be partitioned into sets of associated bimorphisms $\{ \otimes, \otimes_o, \bar{\otimes}, \otimes_\mu \}$ and the bimorphisms in each equivalence class are distinct by Proposition 1.6.4 for part (i) and Proposition 1.6.3 (iv) for part (ii). Hence the result follows.

Theorem 1.6.6 If $(P,\leq)$ is a non-trivial poset such that μ has no fixed points and $S'M(P,\leq)$ is finite then $4 \mid \#(S'M(P,\leq))$.

Proof: From Corollary 1.5.2 (ii) $HM(P,\leq) = \emptyset$ and so $H'S'M(P,\leq) = M(P,\leq) - \emptyset - SM(P,\leq) = S'M(P,\leq)$ and the result follows.

CHAPTER II

ADJOINTS

§2.1 Adjoint Morphisms

Let $f: (P,\leq) \to (Q,\leq)$ and $g: (Q,\leq) \to (P,\leq)$ be morphisms of posets.

Definition 2.1.1 f is a left adjoint to g and g is a right adjoint to f and we write $f \dashv g$ if $f(a) \leq b \iff a \leq g(b)$ for all $a \in P$, $b \in Q$.

It is easily seen that the left adjoint (or alternatively the right adjoint) of a given morphism of posets, if it exists at all, is uniquely determined.

Theorem 2.1.1 Let $g: (P,\leq) \to (Q,\leq)$ be a morphism of posets, then g has a left adjoint f if and only if the set $G_y = \{x \mid g(x) \geq y\}$ has a first element $f(y)$ for all $y \in Q$. [2, p. 491, proposition 4].

Proof: If g has a left adjoint f, then $f(y) \leq x \iff y \leq g(x) \iff x \in G_y$. Hence $f(y) \in G_y$ and $f(y)$ is less than or equal to all other elements of G_y.

Conversely, if each G_y has a first element $f(y)$ then $f(y) \leq x \iff x \in G_y \iff y \leq g(x)$. We must show that f is a morphism of posets. Assume $a \leq b$ for $a, b \in Q$. $G_b = \{x \mid g(x) \geq b\} \subseteq \{x \mid g(x) \geq a\} = G_a$. $f(b) \in G_b$ so $f(b) \in G_a$, therefore $f(b) \geq$ the first element of G_a, namely $f(a)$.

Theorem 2.1.2 Let $f: (P,\leq) \to (Q,\leq)$ be a morphism of posets, then f has a right adjoint g if and only if the set $F_y = \{x \mid f(x) \leq y\}$ has a last element $g(y)$ for all $y \in Q$.

Proof: This is the dual of Theorem 2.1.1.

Proposition 2.1.3 Let $(P,\leq)$ and $(Q,\leq)$ be posets where ω is the last element of P, $\overline{\omega}$ is the last element of Q, and $g : (P,\leq) \to (Q,\leq)$ be a morphism of posets. If g has a left adjoint then $g(\omega) = \overline{\omega}$. Conversely if $(P,\leq)$ is well ordered and $g(\omega) = \overline{\omega}$ then g has a left adjoint.

Proof: => g has a left adjoint implies that $G_{\overline{\omega}} = \{x \mid g(x) \geq \overline{\omega}\}$ is non-empty, i.e. there exists $x \in P$ such that $g(x) \geq \overline{\omega}$, now $\omega \geq x$ so $g(\omega) \geq g(x) \geq \overline{\omega}$, also $g(\omega) \leq \overline{\omega}$ so $g(\omega) = \overline{\omega}$.

<= $g(\omega) \geq y$ for all $y \in Q$, so all sets G_y are non-empty, by the data they have first elements and the result follows from Theorem 2.1.1.

Proposition 2.1.4 Let $(P,\leq)$ and $(Q,\leq)$ be posets such that 0 is the first element of P, $\overline{0}$ the first element of Q, and $f : (P,\leq) \to (Q,\leq)$ be a morphism of posets. If f has a right adjoint then $f(0) = \overline{0}$. Conversely if $(P,\geq)$ is well ordered and $f(0) = \overline{0}$ then f has a right adjoint.

Proof: This is the dual of Proposition 2.1.3.

§2.2 Adjoint Bimorphisms

Let $(P,\leq)$, $(Q,\leq)$ and $(R,\leq)$ be posets and

$\otimes : (P,\leq) \times (Q,\leq) \to (R,\leq)$, $(x,y) \mapsto x \otimes y$, $x \in P$, $y \in Q$, and

$\pitchfork : (Q,\geq) \times (R,\leq) \to (P,\leq)$, $(u,v) \mapsto u \pitchfork v$, $u \in Q$, $v \in R$, be

bimorphisms.

Definition 2.2.1 $\otimes$ will be said to be a <u>left adjoint</u> of $\pitchfork$ and $\pitchfork$ to be a <u>right adjoint</u> of $\otimes$ denoted $\otimes \dashv \pitchfork$ if $a \otimes b \leq c \iff a \leq b \pitchfork c$ for all $a \in P$, $b \in Q$, $c \in R$.

Remark: If $\otimes \dashv \pitchfork$ then $(- \otimes b) \dashv (b \pitchfork -)$ for all choices of $b \in Q$, where $- \otimes b$ and $b \pitchfork -$ are the obvious morphisms of posets.

Theorem 2.2.1 Given a bimorphism $\pitchfork : (Q,\geq) \times (R,\leq) \to (P,\leq)$ assume for each element $b \in Q$ that $b \pitchfork - : (Q \times R,\leq) \to (P,\leq)$ has a left adjoint $- \otimes b : (P \times Q,\leq) \to (R,\leq)$. Then $\pitchfork$ has a left adjoint bimorphism $\otimes : (P,\leq) \times (Q,\leq) \to (R,\leq)$.

Proof: If $- \otimes b$ is a left adjoint of $b \pitchfork -$ then the function $\otimes : P \times Q \to R$ is clearly well defined and if $a \leq a'$ in $(P,\leq)$ then for $b \in Q$, $a \otimes b \leq a' \otimes b$ in $(R,\leq)$ since $- \otimes b$ is a morphism of posets.

Let $b \leq b'$ in $(Q,\leq)$ and $a \in P$ then $a \otimes b$ = the first element of $G_{a \otimes b}$ where $G_{a \otimes b} = \{x | b \pitchfork x \geq a\}$ and $a \otimes b'$ = the first element of $G_{a \otimes b'}$ where $G_{a \otimes b'} = \{x | b' \pitchfork x \geq a\}$. Now $G_{a \otimes b'} \subseteq G_{a \otimes b}$ and $a \otimes b' \in G_{a \otimes b'}$, so $a \otimes b' \in G_{a \otimes b}$ and the first element of $G_{a \otimes b}$ is less than or equal to $a \otimes b'$, i.e. $a \otimes b \leq a \otimes b'$.

Hence $\otimes$ is a bimorphism left adjoint to $\pitchfork$.

Theorem 2.2.2 Given a bimorphism $\otimes : (P,\leq) \times (Q,\leq) \to (R,\leq)$ assume for each element $b \in Q$ that the morphism $- \otimes b : (P \times Q,\leq) \to (R,\leq)$ has a right adjoint morphism $b \pitchfork - : (Q \times R,\leq) \to (P,\leq)$. Then $\otimes$ has a right adjoint bimorphism $\pitchfork : (Q,\geq) \times (R,\leq) \to (P,\leq)$.

Proof: This is similar to that of Theorem 2.2.1.

Theorem 2.2.3 Let $\pitchfork : (Q,\geq) \times (R,\leq) \to (P,\leq)$ be a bimorphism of posets, then $\pitchfork$ has a left adjoint $\otimes : (P,\leq) \times (Q,\leq) \to (R,\leq)$ if and only if the set $G_y^b = \{x \mid b \pitchfork x \geq y\}$ has a first element $y \otimes b$, for all $b \in Q$, $y \in P$.

Proof: It follows from Theorem 2.1.1 that $b \pitchfork -$ has a left adjoint $- \otimes b$; the result then follows from Theorem 2.2.1.

Theorem 2.2.4 Let $\otimes : (P,\leq) \times (Q,\leq) \to (R,\leq)$ be a bimorphism of posets, then $\otimes$ has a right adjoint bimorphism $\pitchfork : (Q,\geq) \times (R,\leq) \to (P,\leq)$ if and only if the set $F_y^b = \{x \mid x \otimes b \leq y\}$ has a last element $b \pitchfork y$, for all $b \in Q$, $y \in P$.

Proof: This is similar to that of Theorem 2.2.3.

Theorem 2.2.5 Let $(P,\leq)$, $(Q,\leq)$, $(R,\leq)$ be posets with ω' the last element of $(P,\leq)$, ω the last element of $(Q,\leq)$, $\bar{\omega}$ the last element of $(R,\leq)$ and $\pitchfork : (Q,\geq) \times (R,\leq) \to (P,\leq)$ be a bimorphism. If $\pitchfork$ has a left adjoint then $\omega \pitchfork \bar{\omega} = \omega'$. Conversely if $(R,\leq)$ is well ordered and $\omega \pitchfork \bar{\omega} = \omega'$, then $\pitchfork$ has a left adjoint.

Proof: If $\pitchfork$ has a left adjoint then $\omega \pitchfork -$ has a left adjoint and $\omega \pitchfork \bar{\omega} = \omega'$ by Proposition 2.1.3.

Conversely, if $b \in (Q,\geq)$ then $b \leq \omega$, so $b \pitchfork \bar{\omega} \geq \omega \pitchfork \bar{\omega} = \omega'$ therefore $b \pitchfork \bar{\omega} = \omega'$, $b \pitchfork -$ has a left adjoint by Proposition 2.1.3 and the result follows by Theorem 2.2.1.

Theorem 2.2.6 Let $(P,\leq)$, $(Q,\leq)$, $(R,\leq)$ be posets with 0 as the first element of $(P,\leq)$, $\bar{0}$ as the first element of $(R,\leq)$, ω as the last element of $(Q,\leq)$ and $\otimes : (P,\leq) \times (Q,\leq) \to (R,\leq)$ be a bimorphism. If $\otimes$ has a right adjoint then $0 \otimes \omega = \bar{0}$. Conversely if $(P,\geq)$ is well ordered and $0 \otimes \omega = \bar{0}$, then $\otimes$ has a right adjoint.

Proof: If $\otimes$ has a right adjoint then $- \otimes \omega$ has a right adjoint and $0 \otimes \omega = \bar{0}$ by Proposition 2.1.4.

Conversely, if $b \in (Q,\leq)$ then $0 \otimes b \leq 0 \otimes \omega = \bar{0}$; it follows by Proposition 2.1.4 that $- \otimes b$ has a right adjoint, and the result follows by Theorem 2.2.2.

CHAPTER III

CLOSED AND MONOIDAL CLOSED POSETS

§3.1 Basic Definitions

Definition 3.1.1 A closed poset $(P, \leq, \pitchfork, k)$ consists of :

(i) a poset $(P, \leq)$;

(ii) a bimorphism $\pitchfork : (P, \geq) \times (P, \leq) \to (P, \leq)$

such that

$$b \pitchfork c \leq (a \pitchfork b) \pitchfork (a \pitchfork c) \quad \text{for all } a,b,c \in P ;$$

(iii) an element $k \in P$ such that

$$a = k \pitchfork a \quad \text{for all } a \in P ;$$

$$k \leq a \pitchfork b \text{ if and only if } a \leq b \quad \text{for all } a,b \in P.$$

Example: Given $(X, \leq, \cdot)$ is an ordered group we can take $(P, \leq)$ to be $(X, \leq)$, $b \pitchfork c = c \cdot b^{-1}$ for all $b,c \in P$ and k to be the identity.

Definition 3.1.2 A monoidal closed poset $(P, \leq, \otimes, \pitchfork, k)$ consists of a monoidal poset $(P, \leq, \otimes, k)$ and a closed poset $(P, \leq, \pitchfork, k)$ such that $(a \otimes b) \pitchfork c = a \pitchfork (b \pitchfork c)$ for all $a,b,c \in P$. [3, p. 555].

Definition 3.1.3 A symmetric monoidal closed poset is a monoidal closed poset in which $a \otimes b = b \otimes a$ for all $a,b \in P$.

Definition 3.1.4 A cartesian closed poset is a monoidal closed poset $(P, \leq, \wedge, \pitchfork, \omega)$ in which $\wedge$ is the meet and ω the last element of P.

Example: A Heyting Algebra is a cartesian closed poset $(P, \leq, \wedge, \pitchfork, \omega)$ such that $(P, \leq)$ is a lattice with a first element (as well as a last element). [4, p. 9].

Definition 3.1.5 Recalling that a bimorphism $\otimes$ determines an associated opposite bimorphism $\otimes_o$, a <u>monoidal biclosed poset</u> (later abbreviated to <u>biclosed poset</u>) is an associated pair of monoidal closed posets $(P, \leq, \otimes, \pitchfork, k)$ and $(P, \leq, \otimes_o, \pitchfork_o, k)$.

Theorem 3.1.1 Given a closed poset $(P, \leq, \pitchfork, k)$ with first element 0 and last element ω then :

(a) $\omega \pitchfork 0 = 0$;

(b) $0 \pitchfork \omega = \omega$;

(c) $0 \pitchfork (\omega \pitchfork \omega) = \omega$;

(d) $0 \pitchfork (\omega \pitchfork (\omega \pitchfork \omega)) = \omega$;

(e) $(0 \pitchfork k) \pitchfork \omega = \omega$;

(f) $\omega \pitchfork \omega = \omega$ or k is incomparable with $\omega \pitchfork (\omega \pitchfork \omega)$.

Proof: (a) $k \leq \omega$ therefore $0 = k \pitchfork 0 \geq \omega \pitchfork 0$

therefore $0 = \omega \pitchfork 0$.

(b) $0 \leq k$ therefore $0 \pitchfork \omega \geq k \pitchfork \omega = \omega$

therefore $0 \pitchfork \omega = \omega$.

(c) $0 \pitchfork \omega \leq (\omega \pitchfork 0) \pitchfork (\omega \pitchfork \omega)$,

$\omega \leq 0 \pitchfork (\omega \pitchfork \omega)$

i.e. $\omega = 0 \pitchfork (\omega \pitchfork \omega)$.

(d) $0 \pitchfork (\omega \pitchfork \omega) \leq (\omega \pitchfork 0) \pitchfork (\omega \pitchfork (\omega \pitchfork \omega))$

$\omega \leq 0 \pitchfork (\omega \pitchfork (\omega \pitchfork \omega))$

i.e. $\omega = 0 \pitchfork (\omega \pitchfork (\omega \pitchfork \omega))$.

(e) $\omega = k \pitchfork \omega \leq (0 \pitchfork k) \pitchfork (0 \pitchfork \omega) = (0 \pitchfork k) \pitchfork \omega$

therefore $(0 \pitchfork k) \pitchfork \omega = \omega$.

(f) Either (i) $k \leq \omega ⫛ (\omega ⫛ \omega)$,

(ii) $k \geq \omega ⫛ (\omega ⫛ \omega)$

or (iii) k and $\omega ⫛ (\omega ⫛ \omega)$ are incomparable.

In case (i) $\omega \leq \omega ⫛ \omega$ i.e. $\omega' = \omega ⫛ \omega$.

In case (ii) $\omega ⫛ (\omega ⫛ \omega) \leq k$ implies

$\omega = 0 ⫛ (\omega ⫛ (\omega ⫛ \omega)) \leq 0 ⫛ k,$

i.e. $\omega \leq 0 ⫛ k$ so $0 ⫛ k = \omega'$.

Hence by (c) $\omega ⫛ \omega = \omega$ and the result is proved.

Corollary 3.1.2 If $(P,\leq)$ is k-comparable then $\omega ⫛ \omega = \omega$.

Proof: This follows from Theorem 3.1.1.

Corollary 3.1.3 If $(P,\leq)$ is well ordered, with first element 0 and last element ω and $(P,\leq,⫛,k)$ is closed then $⫛$ has a left adjoint bimorphism $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$.

Proof: This follows from Corollary 3.1.2 and Theorem 2.2.5.

§3.2 Relations Between the Data

In this section we will uncover redundancies that exist between the data for a monoidal closed poset. We assume throughout this section that

(i) $(P,\leq)$ is a poset ;

(ii) $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$ is a bimorphism ;

(iii) $⫛ : (P,\geq) \times (P,\leq) \to (P,\leq)$ is a bimorphism ;

such that $a \otimes b \leq c$ if and only if $a \leq b ⫛ c$ for all $a,b,c \in P$.

Proposition 3.2.1 The following conditions are equivalent:

(a) $k \otimes a = a$ for all $a \in P$;

(b) $a \leq b$ if and only if $k \leq a \pitchfork b$ for all $a,b \in P$.

Proof: Part (1) Assume (a) is true.

$$a \leq b \iff k \otimes a \leq b \iff k \leq a \pitchfork b.$$

Part (2) Assume (b) is true.

$a \leq b \iff k \leq a \pitchfork b$ i.e. $k \otimes a \leq b$.

Hence $a \leq a \Rightarrow k \otimes a \leq a$

$k \otimes a \leq k \otimes a \Rightarrow a \leq k \otimes a$

Therefore $a = k \otimes a$.

Proposition 3.2.2 The following conditions are equivalent:

(a) $a \otimes k = a$ for all $a \in P$;

(b) $a = k \pitchfork a$ for all $a \in P$.

Proof: Part (1) Assume (a) is true. For all $a \in P$:

$k \pitchfork a \leq k \pitchfork a \Rightarrow (k \pitchfork a) \otimes k \leq a \Rightarrow k \pitchfork a \leq a$.

Also $a \otimes k = a$ implies $a \leq k \pitchfork a$, so $a = k \pitchfork a$.

Part (2) Assume (b) is true. For all $a \in P$:

$a \otimes k \leq a \otimes k \Rightarrow a \leq k \pitchfork (a \otimes k) \Rightarrow a \leq a \otimes k$.

Also $a = k \pitchfork a$ implies $a \otimes k \leq a$, so $a = a \otimes k$.

Proposition 3.2.3 The following conditions are equivalent:

(a) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ for all $a,b,c \in P$;

(b) $(b \otimes c) \pitchfork d = b \pitchfork (c \pitchfork d)$ for all $b,c,d \in P$.

Proof: Part (1) Assume (a) is true.

$$(a \otimes b) \otimes c \leq d \iff a \otimes b \leq c \pitchfork d \iff a \leq b \pitchfork (c \pitchfork d)$$

$$a \otimes (b \otimes c) \leq d \iff a \leq (b \otimes c) \pitchfork d$$

It follows $(b \otimes c) \pitchfork d = b \pitchfork (c \pitchfork d)$.

Part (2) Assume (b) is true.

$$a \leq (b \otimes c) \pitchfork d \iff a \otimes (b \otimes c) \leq d$$

$$a \leq b \pitchfork (c \pitchfork d) \iff a \otimes b \leq c \pitchfork d \iff (a \otimes b) \otimes c \leq d$$

It easily follows that $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

Proposition 3.2.4 The following conditions are equivalent:

(a) $(a \otimes b) \pitchfork c \leq a \pitchfork (b \pitchfork c)$ for all $a,b,c \in P$;

(b) $a \pitchfork c \leq (b \pitchfork a) \pitchfork (b \pitchfork c)$ for all $a,b,c \in P$.

Proof: Part (1) Assume (a) is true.

$b \pitchfork a \leq b \pitchfork a$

$\Rightarrow (b \pitchfork a) \otimes b \leq a$

$\Rightarrow ((b \pitchfork a) \otimes b) \pitchfork c \geq a \pitchfork c$

but $((b \pitchfork a) \otimes b) \pitchfork c \leq (b \pitchfork a) \pitchfork (b \pitchfork c)$ by (a)

therefore $(b \pitchfork a) \pitchfork (b \pitchfork c) \geq a \pitchfork c$.

Part (2) Assume (b) is true.

$a \otimes b \leq a \otimes b$

$\Rightarrow a \leq b \pitchfork (a \otimes b)$

$\Rightarrow a \pitchfork (b \pitchfork c) \geq (b \pitchfork (a \otimes b)) \pitchfork (b \pitchfork c)$

$\Rightarrow a \pitchfork (b \pitchfork c) \geq (a \otimes b) \pitchfork c$ by (b).

Remark: It follows from Proposition 3.2.3 and 3.2.4 that $\otimes$ associative implies $a \pitchfork c \leq (b \pitchfork a) \pitchfork (b \pitchfork c)$ for all $a,b,c \in P$. It is not immediately obvious if the converse holds; however on page 43 we list seven examples to show that the converse is not true in general.

The following result will prove useful in Chapter IV.

Proposition 3.2.5 Given $(P,\leq,\pitchfork,k)$ is a closed poset structure such that 0 is the first element of P, ω is the last element of P, and $(P,\leq)$ is well ordered then there exists a left adjoint $\otimes$ to $\pitchfork$ satisfying:

$$a \otimes k = a$$
$$k \otimes a = a$$
$$0 \otimes \omega = 0.$$

Proof: By Corollary 3.1.3 $\otimes \dashv \pitchfork$. By Theorem 2.2.6 $0 \otimes \omega = 0$. By Proposition 3.2.1 and Proposition 3.2.2 the identity conditions hold for $\otimes$.

§3.3 Expanding Monoidal and Closed Posets

Theorem 3.3.1 A monoidal poset $(P,\leq,\otimes,k)$ is "expandable" into a monoidal closed poset if and only if $\otimes$ has a right adjoint.

Proof: This follows from Propositions 3.2.1, 3.2.2, 3.2.3, 3.2.4.

Corollary 3.3.2 A monoidal poset $(P,\leq,\otimes,k)$ with $(P,\geq)$ well ordered, 0 the first element of P and ω the last element of P, is "expandable" into a monoidal closed poset if and only if $0 \otimes \omega = 0$.

Proof: This follows from Theorems 2.2.6 and 3.3.1.

Theorem 3.3.3 A closed poset $(P,\leq,\pitchfork,k)$ is "expandable" into a monoidal closed poset if and only if

(a) $\pitchfork$ has a left adjoint $\otimes$ and $\otimes$ is associative, or equivalently

(b) $\pitchfork$ has a left adjoint $\otimes$ such that

$$(a \otimes b) \pitchfork c = a \pitchfork (b \pitchfork c) \quad \text{for all } a,b,c \in P.$$

Proof: This follows from Propositions 3.2.1, 3.2.2, 3.2.3.

Remark: We give seven examples on page 43 to show that the $\otimes$ associative condition in (a) is necessary.

Corollary 3.3.4 A closed poset $(P,\leq,\pitchfork,k)$ such that 0 is the first element of P, ω is the last element of P and $(P,\leq)$ is well ordered, is "expandable" into a monoidal closed poset if and only if $\otimes$, the left adjoint of $\pitchfork$ which must exist, is associative.

Proof: This follows from Corollary 3.1.3 and Theorem 3.3.3.

Proposition 3.3.5 If $(P,\leq)$ is a poset with $\wedge, 0, \omega$, such that $(P,\geq)$ is well ordered then $(P,\leq,\wedge,\omega)$ is "expandable" into a cartesian closed poset.

Proof: Since $(P,\leq)$ has a first element 0 and $0 \wedge x = 0$ for all $x \in P$, then given b, $y \in P$ the sets $F_y^b = \{x \in P \mid x \wedge b \leq y\}$ are always non-empty. $(P,\geq)$ is well ordered, so each set F_y^b has a last element. By Theorem 2.2.4 $- \wedge b$ has a right adjoint for all $b \in P$ and the result follows from Corollary 3.3.2.

Proposition 3.3.6 If $(P,\leq)$ is a finite chain with first element 0 and last element ω then a monoidal poset structure $(P,\leq,\otimes,k)$ is "expandable" into a biclosed poset if and only if $0 \otimes \omega = 0 = \omega \otimes 0$.

Proof: It follows from the data that $0 \otimes_0 \omega = 0$ and by Proposition 1.3.1 $(P,\leq,\otimes_0,k)$ is a monoidal poset and the result follows from Corollary 3.3.2.

Proposition 3.3.7 If $(P,\leq)$ is a finite chain with first element 0 and last element ω, then a closed poset structure $(P,\leq,\pitchfork,k)$ is "expandable" into a biclosed poset if and only if

(i) $\otimes$; the left adjoint of $\pitchfork$, is associative ;

(ii) $0 \pitchfork 0 = \omega$.

Proof: By Corollary 3.1.3 $\pitchfork$ has a left adjoint $\otimes$. By Theorem 3.3.3 $(P,\leq,\otimes,\pitchfork,k)$ is monoidal closed. By Proposition 1.3.1 $(P,\leq,\otimes_0,k)$ is a monoidal poset. If $0 \pitchfork 0 = \omega$ then by Theorem 2.2.3 $\omega \otimes 0 = \min \{x \in P \mid 0 \pitchfork x \geq \omega\} = 0$. Hence $0 \otimes_0 \omega = 0$. By Corollary 3.3.2 $(P,\leq,\otimes_0,k)$ is expandable into a monoidal closed poset and the result follows.

§3.4 Relations Between $M(P,\leq)$, $SM(P,\leq)$, $MC(P,\leq)$, $SMC(P,\leq)$

<u>Throughout this section we assume that the poset $(P,\leq)$ under consideration has a given inversion μ.</u>

Lemma 3.4.1 Let $(P,\leq)$ have first element 0 and last element ω.

(a) There exists a bijection between the set of monoidal posets such that $0 \otimes \omega = 0$, i.e. $M(P,\leq ; 0 \otimes \omega = 0)$ and the set of monoidal posets such that $0 \otimes \omega = \omega$ i.e. $M(P,\leq ; 0 \otimes \omega = \omega)$.

(b) There exists a bijection between the set of symmetric monoidal posets such that $0 \otimes \omega = 0$ i.e. $SM(P,\leq ; 0 \otimes \omega = 0)$ and the set of symmetric monoidal posets such that $0 \otimes \omega = \omega$ i.e. $SM(P,\leq ; 0 \otimes \omega = \omega)$.

Proof: (a) We can define functions $\mu_1 : M(P,\leq ; 0 \otimes \omega = 0) \to M(P,\leq ; 0 \otimes \omega = \omega)$ by $\mu_1(P,\leq,\otimes,k) = (P,\leq,\overline{\otimes},\mu k)$, and $\mu_2 : M(P,\leq ; 0 \otimes \omega = \omega) \to M(P,\leq ; 0 \otimes \omega = 0)$ by $\mu_2(P,\leq,\otimes,k) = (P,\leq,\overline{\otimes},\mu k)$. We see that $\mu_1\mu_2$ and $\mu_2\mu_1$ are identities and the result follows.

(b) In a similar manner it may be shown that there is a bijection between $SM(P,\leq ; 0 \otimes \omega = 0)$ and $SM(P,\leq ; 0 \otimes \omega = \omega)$, since $\otimes$ is commutative if and only if $\overline{\otimes}$ is commutative.

Notation : We will use $MC'(P,\leq)$ to denote all monoidal structures on the poset $(P,\leq)$ that are not closed. $SMC'(P,\leq)$, $S'MC(P,\leq)$, $S'MC'(P,\leq)$ and $SMC(P,\leq)$ will have the obvious meanings.

Theorem 3.4.2 Let $(P,\leq)$ be a non-trivial chain with first element 0 and last element ω then the rule $\otimes \to \overline{\otimes}$ determines bijections:

(i) $MC(P,\leq) \to MC'(P,\leq)$

(ii) $SMC(P,\leq) \to SMC'(P,\leq)$

(iii) $S'MC(P,\leq) \to S'MC'(P,\leq)$.

Proof: (i) $MC(P,\leq) = M(P,\leq ; 0 \otimes \omega = 0)$ by Theorem 2.2.6, $MC'(P,\leq) = M(P,\leq ; 0 \otimes \omega = \omega)$ by Theorem 2.2.6 and Corollary 1.2.2;

the result is then immediate from Lemma 3.4.1 (a).

(ii) follows in a similar manner using Lemma 3.4.1 (b).

(iii) follows from (i) and (ii) since $MC(P,\leq)$ is the disjoint union of $SMC(P,\leq)$ and $S'MC(P,\leq)$, and $MC'(P,\leq)$ is the disjoint union of $SMC'(P,\leq)$ and $S'MC'(P,\leq)$.

Corollary 3.4.3 (i) $\#(M(P,\leq)) = 2 \times \#(MC(P,\leq)) = 2 \times \#(MC'(P,\leq))$

(ii) $\#(SM(P,\leq)) = 2 \times \#(SMC(P,\leq)) = 2 \times \#(SMC'(P,\leq))$.

Proof: This follows immediately from Theorem 3.4.2 parts (i) and (ii) respectively since $M(P,\leq)$ is the disjoint union of $MC(P,\leq)$ and $MC'(P,\leq)$, and $SM(P,\leq)$ is the disjoint union of $SMC(P,\leq)$ and $SMC'(P,\leq)$.

Corollary 3.4.4 $\#(M(P,\leq)) \leq 2 \times \#(C(P,\leq))$.

Proof: $\frac{1}{2} \times \#(M(P,\leq)) = \#(MC(P,\leq))$ by Corollary 3.4.3 (i).

$\leq \#(C(P,\leq))$ since $MC(P,\leq) \subseteq C(P,\leq)$.

Corollary 3.4.5 If μ has no fixed points and $S'M(P,\leq)$ is finite (e.g. if $(P,\leq)$ is a finite chain with an even number of elements) then $2 \mid \#(S'MC(P,\leq))$ and $2 \mid \#(S'MC'(P,\leq))$.

Proof: $S'M(P,\leq)$ is the disjoint union of $S'MC(P,\leq)$ and $S'MC'(P,\leq)$. By Theorem 1.6.6 $4 \mid \#(S'M(P,\leq))$. By Theorem 3.4.2 (iii) $\#(S'MC(P,\leq)) = \#(S'MC'(P,\leq))$. Therefore $4 \mid 2 \times \#(S'MC(P,\leq))$, $4 \mid 2 \times \#(S'MC'(P,\leq))$ and the result follows.

Under weaker assumptions the results of Corollary 3.4.3 hold "in one direction".

Theorem 3.4.6 Let $(P,\leq)$ be a non-trivial poset with first element 0 and last element ω then

(i) $2 \times \#(MC(P,\leq)) \leq \#(M(P,\leq))$

(ii) $2 \times \#(SMC(P,\leq)) \leq \#(SM(P,\leq))$.

Proof: (i) $2 \times \#(MC(P,\leq)) \leq 2 \times \#(M(P,\leq\ ;\ 0 \otimes \omega = 0))$ by Theorem 2.2.6,

$= \#(M(P,\leq\ ;\ 0 \otimes \omega = 0)) \cup \#(M(P,\leq\ ;\ 0 \otimes \omega = \omega))$

by Lemma 3.4.1,

$\leq \#(M(P,\leq))$ by Theorem 1.2.1.

(ii) A similar proof applies to the symmetric monoidal case.

CHAPTER IV

CONSTRUCTION OF MONOIDAL AND CLOSED STRUCTURES ON A CHAIN $(P,\leq)$

§4.1 Procedure

We will describe a procedure for listing all possible monoidal and closed structures on a finite chain $(P,\leq)$.

STEP I

If $(P,\leq,\otimes,k)$ is a monoidal finite chain then

(i) $\otimes : (P,\leq) \times (P,\leq) \to (P,\leq)$ is a bimorphism ;

(ii) $a \otimes k = a$ and $k \otimes a = a$ for all $a \in P$;

(iii) $0 \otimes \omega = 0$ or ω (Corollary 1.2.2) ;

(iv) $\omega \otimes 0 = 0$ or ω (Corollary 1.2.2).

For each value of k in turn we draw up all possible monoid tables for $\otimes$ on P, such that conditions (i), (ii), (iv), are satisfied and $0 \otimes \omega = 0$. If the elements of P are written down in increasing order (i) simply asserts that as we move along (from left to right) the rows of these tables, or down their columns, the sequence of terms is non-decreasing. We do not need to consider the cases where $0 \otimes \omega = \omega$ at this stage, as such cases will be later obtained from the above cases by means of the rule $\otimes \mapsto \bar{\otimes}$ (see Corollary 1.2.2, Lemma 1.4.2, and Theorem 3.4.2).

STEP II

To compute $MC(P,\leq)$ we check all bimorphisms $\otimes$ in Step I for associativity. By Corollary 3.3.2 each of these monoidal structures thus determined is expandable into a monoidal closed poset. Hence for each bimorphism $\otimes$ we can write down its corresponding right adjoint $\pitchfork$ by the rule

$$b \pitchfork a = \max \{x \in P \mid x \otimes b \leq a\} \text{ for all } a,b \in P$$

(see Theorem 2.2.4). Thus we have determined all monoidal closed structures on $(P,\leq)$.

STEP III

To determine $SMC(P,\leq)$ we check each bimorphism $\otimes$ satisfying Step II for commutativity.

STEP IV

To compute $MC'(P,\leq)$ we apply the bijection $\otimes \longmapsto \bar{\otimes}$ to the results of Step II. We thus obtain the monoidal cases where $0 \otimes \omega = \omega$ as explained at the end of Step I.

STEP V

$M(P,\leq)$ is obtained as $MC(P,\leq) \cup MC'(P,\leq)$.

STEP VI

$SMC'(P,\leq)$ and $SM(P,\leq)$ are then easily obtained by listing the commutative $\otimes$'s in $MC'(P,\leq)$ and $M(P,\leq)$.

STEP VII

By Proposition 3.2.5 given a closed poset structure $(P,\leq,\pitchfork,k)$ on our chain $(P,\leq)$, then there exists $\otimes$ satisfying the identity conditions and $0 \otimes \omega = 0$. These $\otimes$'s have already been listed in Step I, and the subset of these appearing in Step II are monoidal closed and certainly closed. Hence to find $M'C(P,\leq)$ we use Theorem 2.2.4 to list the $\pitchfork$'s corresponding to the non-associative $\otimes$'s of Step I, (i.e. those which were "rejected" in Step II). These $\pitchfork$'s are then checked to see if the condition $a \pitchfork b \leq (c \pitchfork a) \pitchfork (c \pitchfork b)$ for all $a,b,c \in P$, is satisfied, those which do are the $\pitchfork$'s of $M'C(P,\leq)$.

STEP VIII

$C(P,\leq)$ is then known, as it is just $M'C(P,\leq) \cup MC(P,\leq)$.

§4.2 Example of the Technique Used

We illustrate the above method, used in obtaining these structures, by showing how to get all closed, monoidal, monoidal closed, symmetric monoidal, and symmetric monoidal closed structures on $(\{0,1,2,3\}, \leq)$ with the identity element $k = 2$.

STEP I

Since the identity element is 2, $a \otimes 2 = a = 2 \otimes a$ for all a.

Hence we need to fill the following $\otimes$ - tables according to the definition of $\otimes$:

$\otimes$	0	1	2	3	
0	0	0	0	$\boxed{0}$	← This follows from Step I in §4.1
1	0		1		
2	0	1	2	3	
3			3	3	

By Corollary 1.2.2, $3 \otimes 0 = 0$ or 3, hence two possible situations arise:

Case (i)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0		1	
2	0	1	2	3
3	0		3	3

This could be 1,2, or 3

This could be 0 or 1

This could be 1,2, or 3

18 cases altogether

Case (ii)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0		1	
2	0	1	2	3
3	3	3	3	3

This could be 1,2, or 3

This could be 0 or 1

6 cases altogether

STEP II

To compute $MC(P,\leq)$ each of these 24 cases found in Step I has to be checked by hand for associativity. This is easier than it seems for $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ in the first 18 cases if any one of either a,b,c is either 2 = the identity (easily verified) or 0 (for then $(a \otimes b) \otimes c = a \otimes (b \otimes c) = 0$). Hence the only cases that really need checking involve a,b,c being selected from $\{1,3\}$ i.e.

$(1 \otimes 1) \otimes 1$ $\quad$ $(1 \otimes 3) \otimes 3$ $\quad$ $(3 \otimes 3) \otimes 1$

$(1 \otimes 1) \otimes 3$ $\quad$ $(3 \otimes 1) \otimes 1$ $\quad$ $(3 \otimes 3) \otimes 3$

$(1 \otimes 3) \otimes 1$ $\quad$ $(3 \otimes 1) \otimes 3$ $\quad$ i.e. 8 cases.

Now each of these 24 cases has an associated $\pitchfork$ and those cases which are monoidal structures are closed structures by Corollary 3.3.2. Of these 24 cases the following are monoidal closed structures.

(1)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	2	2	2	3
1	0	1	1	3	1	0	2	2	3
2	0	1	2	3	2	0	1	2	3
3	3	3	3	3	3	0	0	0	3

(2)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	3	3	3	3
1	0	1	1	3	1	0	3	3	3
2	0	1	2	3	2	0	1	2	3
3	0	1	3	3	3	0	0	0	3

(3)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	3	3	3	3
1	0	1	1	1	1	0	2	2	3
2	0	1	2	3	2	0	1	2	3
3	0	3	3	3	3	0	1	1	3

(4)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	3	3	3	3
1	0	1	1	3	1	0	2	2	3
2	0	1	2	3	2	0	1	2	3
3	0	3	3	3	3	0	0	0	3

(5)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	3	3	3	3
1	0	1	1	1	1	0	3	3	3
2	0	1	2	3	2	0	1	2	3
3	0	1	3	3	3	0	1	1	3

(6)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	3	3	3	3
1	0	0	1	1	1	1	3	3	3
2	0	1	2	3	2	0	1	2	3
3	0	1	3	3	3	0	1	1	3

STEP III

To compute $SMC(P,\leq)$ we check each bimorphism $\otimes$ satisfying Step II for commutativity. The examples in Step II satisfying Step III are (4),(5),(6).

STEP IV

To compute $MC'(P,\leq)$ we apply the bijection $\otimes \mapsto \bar{\otimes}$ to the results of Step II. Thus we have the following tables satisfying Step IV:

(7)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

(8)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	2	2	2	3
3	3	3	3	3

(9)

$\otimes$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	0	2	2	3
3	3	3	3	3

(10)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	0	2	2	3
3	3	3	3	3

(11)

$\otimes$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	2	3
3	3	3	3	3

(12)

$\otimes$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	3	3
3	3	3	3	3

STEP V

Therefore examples (1) through (12) belong to $M(P,\leq)$.

STEP VI

Examples (10), (11), (12) belong to $SMC'(P,\leq)$ and examples (4), (5), (6), (10), (11), (12) belong to $SM(P,\leq)$.

STEP VII

To compute $M'C(P,\leq)$.

Of the twenty-four $\otimes$'s in Step I, six were found to be associative in Step II. Hence we have to write down the $\pitchfork$'s corresponding to the eighteen remaining $\otimes$'s of Step I and check to see if

$$a \pitchfork b \leq (c \pitchfork a) \pitchfork (c \pitchfork b) \quad \text{for all } a,b,c.$$

This holds in the following six cases, so they are the examples belonging to $M'C(P,\leq)$.

(13)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	2	3	3	3
1	0	0	1	1	1	1	3	3	3
2	0	1	2	3	2	0	1	2	3
3	1	1	3	3	3	0	1	1	3

(14)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	2	2	2	3
1	0	0	1	1	1	1	2	2	3
2	0	1	2	3	2	0	1	2	3
3	3	3	3	3	3	0	1	1	3

(15)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	2	2	2	3
1	0	0	1	3	1	1	2	2	3
2	0	1	2	3	2	0	1	2	3
3	3	3	3	3	3	0	0	0	3

(16)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	2	3	3	3
1	0	1	1	1	1	0	3	3	3
2	0	1	2	3	2	0	1	2	3
3	1	1	3	3	3	0	1	1	3

(17)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	2	3	3	3
1	0	1	1	1	1	0	2	2	3
2	0	1	2	3	2	0	1	2	3
3	1	3	3	3	3	0	1	1	3

(18)

$\otimes$	0	1	2	3	$\pitchfork$	0	1	2	3
0	0	0	0	0	0	2	2	2	3
1	0	1	1	1	1	0	2	2	3
2	0	1	2	3	2	0	1	2	3
3	3	3	3	3	3	0	1	1	3

<u>STEP VIII</u>

$C(P,\leq)$ consists of the examples listed in Step II for $MC(P,\leq)$, i.e. examples (1) to (6) plus the examples listed in Step VII for $M'C(P,\leq)$, i.e. examples (13) to (18).

§4.3 Complete List of Monoidal and Closed Structures on the Ordinal Numbers 2,3,4.

The following lists can be obtained by applying the techniques of §5.1, as illustrated in §5.2. The identity elements are not listed separately as they can easily be picked out of the $\otimes$ tables.

I $(P,\leq) = (\{0,1\},\leq)$ = the ordinal number 2.

SYMMETRIC MONOIDAL POSET $SMC'(P,\leq)$.

(1)

$\otimes$	0	1
0	0	1
1	1	1

SYMMETRIC MONOIDAL CLOSED POSET SMC(P,$\leq$)

(2)

$\otimes$	0	1
0	0	0
1	0	1

$\pitchfork$	0	1
0	1	1
1	0	1

II (P,$\leq$) = ({0,1,2},$\leq$) = the ordinal number 3.

MONOIDAL POSET S'MC'(P,$\leq$)

(1)

$\otimes$	0	1	2
0	0	0	2
1	0	1	2
2	0	2	2

MONOIDAL CLOSED POSET S'MC(P,$\leq$)

(2)

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	2	2	2

$\pitchfork$	0	1	2
0	1	1	2
1	0	1	2
2	0	0	2

SYMMETRIC MONOIDAL POSETS SMC'(P,$\leq$)

(3)

$\otimes$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

(4)

$\otimes$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

(5)

$\otimes$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

SYMMETRIC MONOIDAL CLOSED POSETS SMC(P,$\leq$)

(6)

$\otimes$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\pitchfork$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

(7)

$\otimes$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\pitchfork$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

(8)

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\pitchfork$	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

III (P,$\leq$) = ({0,1,2,3},$\leq$) = the ordinal number 4.

CLOSED POSETS M'C(P,$\leq$)

The seven closed posets below are non-monoidal because the $\otimes$ in each case is not associative. For each of the seven cases an example showing that $\otimes$ is not associative is given.

(1)

$\otimes$	0	1	2	3	$\multimap$	0	1	2	3
0	0	0	0	0	0	2	3	3	3
1	0	0	1	1	1	1	3	3	3
2	0	1	2	3	2	0	1	2	3
3	1	1	3	3	3	0	1	1	3

$(3 \otimes 0) \otimes 0 \neq 3 \otimes (0 \otimes 0)$

(2)

$\otimes$	0	1	2	3	$\multimap$	0	1	2	3
0	0	0	0	0	0	2	2	2	3
1	0	0	1	1	1	1	2	2	3
2	0	1	2	3	2	0	1	2	3
3	3	3	3	3	3	0	1	1	3

$(1 \otimes 3) \otimes 0 \neq 1 \otimes (3 \otimes 0)$

(3)

$\otimes$	0	1	2	3	$\multimap$	0	1	2	3
0	0	0	0	0	0	2	2	2	3
1	0	0	1	3	1	1	2	2	3
2	0	1	2	3	2	0	1	2	3
3	3	3	3	3	3	0	0	0	3

$(1 \otimes 1) \otimes 3 \neq 1 \otimes (1 \otimes 3)$

(4)

$\otimes$	0	1	2	3	$\multimap$	0	1	2	3
0	0	0	0	0	0	2	3	3	3
1	0	1	1	1	1	0	3	3	3
2	0	1	2	3	2	0	1	2	3
3	1	1	3	3	3	0	1	1	3

$(3 \otimes 0) \otimes 0 \neq 3 \otimes (0 \otimes 0)$

(5)

$\otimes$	0	1	2	3	$\multimap$	0	1	2	3
0	0	0	0	0	0	2	3	3	3
1	0	1	1	1	1	0	2	2	3
2	0	1	2	3	2	0	1	2	3
3	1	3	3	3	3	0	1	1	3

$(3 \otimes 0) \otimes 0 \neq 3 \otimes (0 \otimes 0)$

(6)

$\otimes$	0	1	2	3	$\multimap$	0	1	2	3
0	0	0	0	0	0	2	2	2	3
1	0	1	1	1	1	0	2	2	3
2	0	1	2	3	2	0	1	2	3
3	3	3	3	3	3	0	1	1	3

$(1 \otimes 3) \otimes 0 \neq 1 \otimes (3 \otimes 0)$

(7)

$\otimes$	0	1	2	3	$\multimap$	0	1	2	3
0	0	0	0	0	0	3	3	3	3
1	0	0	0	1	1	1	3	3	3
2	0	1	1	2	2	1	2	3	3
3	0	1	2	3	3	0	1	2	3

$(3 \otimes 1) \otimes 3 \neq 3 \otimes (1 \otimes 3)$

MONOIDAL POSETS S'MC'$(P, \leq)$

(8)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	3	3	3
3	3	3	3	3

(9)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	2	3	3
3	3	3	3	3

(10)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

(11)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	2	2	2	3
3	3	3	3	3

(12)

$\otimes$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	0	2	2	3
3	3	3	3	3

(13)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	1	3
2	0	1	2	3
3	0	3	3	3

MONOIDAL CLOSED POSETS $S'MC(P,\leq)$

(14)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	0	2	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	0	1	3	3
3	0	1	2	3

(15)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	1	2	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	1	3	3	3
2	1	1	3	3
3	0	1	2	3

(16)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	3
2	0	1	2	3
3	3	3	3	3

$\pitchfork$	0	1	2	3
0	2	2	2	3
1	0	2	2	3
2	0	1	2	3
3	0	0	0	3

(17)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	3
2	0	1	2	3
3	0	1	3	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	1	2	3
3	0	0	0	3

(18)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	3
3	0	3	3	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	2	2	3
2	0	1	2	3
3	0	1	1	3

(19)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	3	3	3	3

$\pitchfork$	0	1	2	3
0	2	2	2	3
1	0	1	2	3
2	0	0	2	3
3	0	0	0	3

SYMMETRIC MONOIDAL POSETS $SMC'(P,\leq)$

(20)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

(21)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	3	3
3	3	3	3	3

(22)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	3	3
3	3	3	3	3

(23)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	2	2	3
2	2	2	2	3
3	3	3	3	3

(24)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	2	3	3
2	2	3	3	3
3	3	3	3	3

(25)

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	3	3	3
2	2	3	3	3
3	3	3	3	3

(26)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	0	2	2	3
3	3	3	3	3

(27)

$\otimes$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	2	3
3	3	3	3	3

(28)

$\otimes$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	3	3
3	3	3	3	3

(29)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	0	1	3
2	0	1	2	3
3	3	3	3	3

(30)

$\otimes$	0	1	2	3
0	0	0	0	3
1	0	1	1	3
2	0	1	2	3
3	3	3	3	3

SYMMETRIC MONOIDAL CLOSED POSETS : SMC$(P,\leq)$

(31)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	1	3	3
3	0	1	2	3

(32)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	[illegible]	3	3	3
2	0	1	3	3
3	0	1	2	3

(33)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	2	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	1	1	3	3
3	0	1	2	3

(34)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	2	3	3
3	0	1	2	3

(35)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	1	2	3	3
3	0	1	2	3

(36)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	0	2
3	0	1	2	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	2	2	3	3
3	0	1	2	3

(37)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	3
2	0	1	2	3
3	0	3	3	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	2	2	3
2	0	1	2	3
3	0	0	0	3

(38)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	3
3	0	1	3	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	1	2	3
3	0	1	1	3

(39)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	3
3	0	1	3	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	1	3	3	3
2	0	1	2	3
3	0	1	1	3

(40)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	1	3
3	0	0	0	3

(41)

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

$\pitchfork$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	2	3
3	0	0	0	3

BICLOSED POSETS

It is obvious that symmetric monoidal closed posets are biclosed. However (14), (15), (17), (18), are examples of biclosed posets that are not symmetric monoidal closed posets.

Our results are summarized in the following charts which give the numbers of distinct structures that can be carried by each ordinal number considered. We use the notation $MBC(P,\leq)$ for the set of all biclosed structures on the poset $(P,\leq)$.

TABLE 1 $(P,\leq) = (\{0,1\},\leq)$

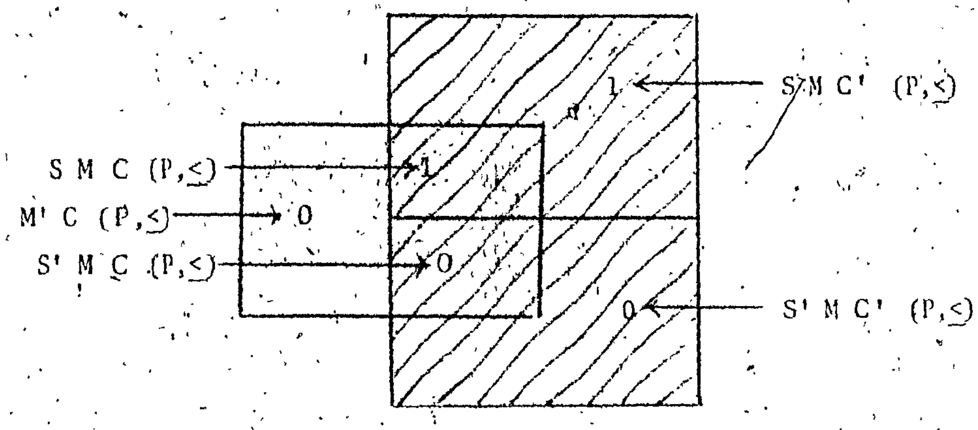


TABLE 2 $(P,\leq) = (\{0,1,2\},\leq)$

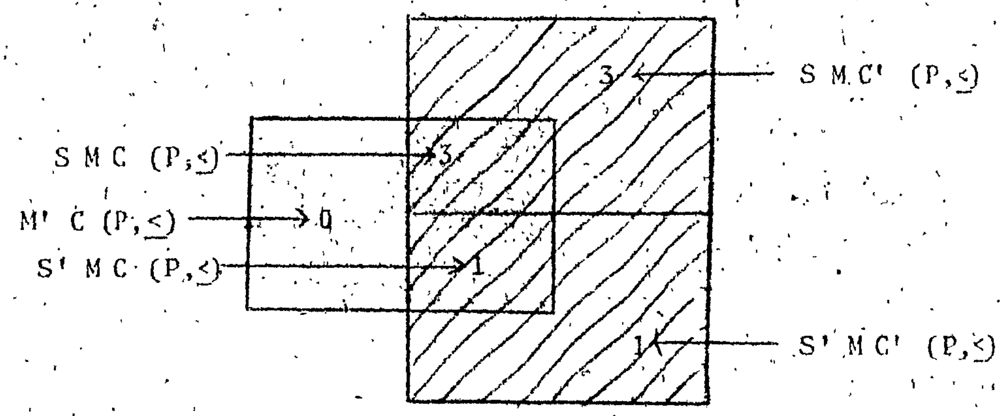
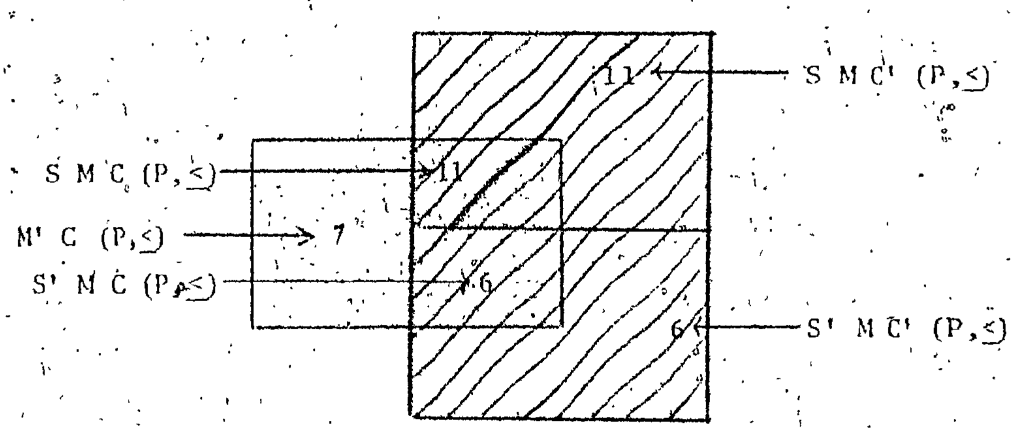


TABLE 3 $(P,\leq) = (\{0,1,2,3\},\leq)$



Remark: The numbers listed in the shaded box in each diagram represent closed structures, whereas the numbers in the stroked boxes represent monoidal structures.

The following table gives the number of elements in the sets $M(P,\leq)$, $C(P,\leq)$, $SM(P,\leq)$, $MC(P,\leq)$, $SMC(P,\leq)$, $MBC(P,\leq)$, and $M'C(P,\leq)$, where $(P,\leq)$ is one of the ordinal numbers 2, 3, 4.

$(P,\leq)$	$M(P,\leq)$	$C(P,\leq)$	$SM(P,\leq)$	$MC(P,\leq)$	$SMC(P,\leq)$	$MBC(P,\leq)$	$M'C(P,\leq)$
$(\{0,1\},\leq)$	2	1	2	1	1	1	0
$(\{0,1,2\},\leq)$	8	4	6	4	3	3	0
$(\{0,1,2,3\},\leq)$	34	24	22	17	11	15	7

The reader will notice that in each case $\#(M(P,\leq)) = 2(\#(MC(P,\leq))$ and $\#(SM(P,\leq)) = 2(\#(SMC(P,\leq))$ (Corollary 3.4.3) and also the condition that 4 divides $\{\#(M(P,\leq)) - \#(SM(P,\leq))\}$ for the ordinal numbers 2 and 4 (Theorem 1.6.6). The examples for the ordinal number 3 show that $\{\#(M(P,\leq)) - \#(SM(P,\leq))\}$ is not always divisible by 4.

CHAPTER V

MONOIDAL AND CLOSED CATEGORIES

§5.1 Monoidal Categories

We assume the standard concepts of category, opposite = dual category, product of categories, product of objects in a category, isomorphism of objects in a category, covariant functor, contravariant functor, bifunctor, adjoint functors, natural transformation, and natural equivalence (denoted by $\simeq$).

Definition 5.1.1 [3, p. 471] A monoidal category $\bar{C} = (C, \otimes, K, r, \ell, a)$ consists of the following six data:

(i) a category C ;

(ii) a bifunctor $\otimes : C \times C \to C$;

(iii) an object K of C ;

(iv) a natural equivalence $r = r_A : A \otimes K \to A$;

(v) a natural equivalence $\ell = \ell_A : K \otimes A \to A$;

(vi) a natural equivalence $a = a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.

These data are to satisfy the following five axioms:

MC1 The following diagram commutes:

$$\begin{tikzcd}
(K \otimes A) \otimes B \arrow[rr, "a_{KAB}"] \arrow[dr, "\ell_A \otimes 1_B"'] & & K \otimes (A \otimes B) \arrow[dl, "\ell_{A \otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

MC2 The following diagram commutes:

$$\begin{tikzcd}
(A \otimes K) \otimes B \arrow[rr, "a_{AKB}"] \arrow[dr, "r_A \otimes 1_B"'] & & A \otimes (K \otimes B) \arrow[dl, "1_A \otimes \ell_B"] \\
& A \otimes B &
\end{tikzcd}$$

MC3 The following diagram commutes:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a_{(A \otimes B)CD}"] \arrow[d, "a_{ABC} \otimes 1_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{AB(C \otimes D)}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A(B \otimes C)D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1_A \otimes a_{BCD}"']
\end{tikzcd}$$

MC4 The following diagram commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes K \arrow[rr, "a_{ABK}"] \arrow[dr, "r_{A \otimes B}"'] & & A \otimes (B \otimes K) \arrow[dl, "1_A \otimes r_B"] \\
& A \otimes B &
\end{tikzcd}$$

MC5 $\ell_K = r_K : K \otimes K \to K$.

Definition 5.1.2 [3, p. 512] A monoidal category $\overline{C}$ together with a symmetry c for $\overline{C}$ is called a <u>symmetric monoidal category.</u> A symmetry for a monoidal category $\overline{C}$ consists of a natural equivalence $c = c_{AB} : A \otimes B \to B \otimes A$ in $\overline{C}$ satisfying the following axioms:

MC6 $c_{BA}c_{AB} = 1 : A \otimes B \to A \otimes B$.

MC7 The following diagram commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a_{ABC}"] \arrow[d, "c_{AB} \otimes 1_C"'] & A \otimes (B \otimes C) \arrow[r, "c_{A(B \otimes C)}"] & (B \otimes C) \otimes A \arrow[d, "a_{BCA}"] \\
(B \otimes A) \otimes C \arrow[r, "a_{BAC}"'] & B \otimes (A \otimes C) \arrow[r, "1_B \otimes c_{AC}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

§5.2 Closed Categories

Definition 5.2.1 [3,p. 428] A <u>closed category</u> $\overline{C} = (C, \pitchfork, K, i, j, L^A_{BC})$ consists of the following data:

(i) a category C ;

(ii) a bifunctor $\pitchfork : C^{op} \times C \to C$;

(iii) an object $K \in C$;

(iv) a natural equivalence $i = i_A : A \to K \pitchfork A$;

(v) a natural transformation $j = j_A : K \to A \pitchfork A$;

(vi) a natural transformation $L = L^A_{BC} : B \pitchfork C \to (A \pitchfork B) \pitchfork (A \pitchfork C)$.

These data are to satisfy the following axioms:

CC1 If F is the functor $\hom(K,-) : C \to \mathrm{Set}$ then $F(A \pitchfork B)$ is naturally equivalent to $\hom(A,B)$ i.e. $\hom(K, A \pitchfork B)$ is naturally equivalent to $\hom(A,B)$.

CC2 The following diagram commutes:

$$\begin{tikzcd}
B \pitchfork B \arrow[rr, "L^A_{BB}"] & & (A \pitchfork B) \pitchfork (A \pitchfork B) \\
& K \arrow[ul, "j_B"] \arrow[ur, "j_{A \pitchfork B}"'] &
\end{tikzcd}$$

CC3 The following diagram commutes:

$$\begin{tikzcd}
A \pitchfork C \arrow[rr, "L^A_{AC}"] \arrow[dr, "i_{A \pitchfork C}"'] & & (A \pitchfork A) \pitchfork (A \pitchfork C) \arrow[dl, "j_A \pitchfork 1_{A \pitchfork C}"] \\
& K \pitchfork (A \pitchfork C) &
\end{tikzcd}$$

CC4 The following diagram commutes:

$$\begin{tikzcd}
C \pitchfork D \arrow[rr, "L^B_{CD}"] \arrow[d, "L^A_{CD}"'] & & (B \pitchfork C) \pitchfork (B \pitchfork D) \arrow[dd, "1_{B \pitchfork C} \pitchfork L^A_{BD}"] \\
(A \pitchfork C) \pitchfork (A \pitchfork D) \arrow[d, "L^{A \pitchfork B}_{A \pitchfork C, A \pitchfork D}"'] & & \\
((A \pitchfork B) \pitchfork (A \pitchfork C)) \pitchfork ((A \pitchfork B) \pitchfork (A \pitchfork D)) \arrow[rr, "L^A_{BC} \pitchfork 1_{(A \pitchfork B) \pitchfork (A \pitchfork D)}"'] & & (B \pitchfork C) \pitchfork ((A \pitchfork B) \pitchfork (A \pitchfork D))
\end{tikzcd}$$

CC5 The following diagram commutes:

$$\begin{tikzcd}
B \pitchfork C \arrow[rr, "L^K_{BC}"] \arrow[dr, "1_B \pitchfork i_C"'] & & (K \pitchfork B) \pitchfork (K \pitchfork C) \arrow[dl, "i_B \pitchfork 1_{K \pitchfork C}"] \\
& B \pitchfork (K \pitchfork C) &
\end{tikzcd}$$

CC6 The map $Fi_{A \pitchfork A} : F(A \pitchfork A) \to F(K \pitchfork (A \pitchfork A))$ which by CC1 may also be written as $Fi_{A \pitchfork A} : \hom(A,A) \to \hom(K, A \pitchfork A)$ sends $1_A \in \hom(A,A)$ to $j_A \in \hom(K, A \pitchfork A)$.

Remark: The above definition is a slight modification of the definition of closed category given by Eilenberg and Kelly [3,p. 428]. Their definition assumes our data (i) - (vi) and our axioms CC2 - CC6 as well as an additional piece of data, namely the existence of a functor $F : C \to Set$ and an axiom such that the following diagram of functors commutes:

$$\begin{tikzcd} C^{op} \times C \arrow[rr, "\pitchfork"] \arrow[dr, "hom"'] & & C \arrow[dl, "F"] \\ & Set & \end{tikzcd}$$

(without relating F to $hom(K,-)$).

It is clear that if we adopt the definition of Eilenberg and Kelly then $F(-)$ is naturally equivalent to $F(K \pitchfork -)$ (by data (iv)) $= hom(K,-)$ (by the above diagram). Hence the definitions are essentially equivalent; our form of the definition was chosen because it modifies more easily to the poset case.

§5.3 Monoidal Closed Categories

Definition 5.3.1 [3,p. 475] A <u>monoidal closed category</u> = $(C, \otimes, \pitchfork, K, i, j, \ell, r, a, L^{A}_{BC})$ consists of the following data:

(i) a monoidal category $(C, \otimes, K, r, \ell, a)$;

(ii) a closed category $(C, \pitchfork, K, i, j, L^{A}_{BC})$;

(iii) a natural equivalence $p = p_{ABC} : (A \otimes B) \pitchfork C \to A \pitchfork (B \pitchfork C)$.

These data are to satisfy the following axioms:

MCC1 The following diagram commutes:

$$\begin{tikzcd}
(K \otimes A) \pitchfork B \arrow[rr, "p_{KAB}"] & & K \pitchfork (A \pitchfork B) \\
& A \pitchfork B \arrow[ul, "\ell_A \pitchfork 1"] \arrow[ur, "i_{A \pitchfork B}"'] &
\end{tikzcd}$$

MCC2 The following diagram commutes:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \pitchfork D \arrow[r, "p_{(A \otimes B)CD}"] & (A \otimes B) \pitchfork (C \pitchfork D) \arrow[r, "p_{AB(C \pitchfork D)}"] & A \pitchfork (B \pitchfork (C \pitchfork D)) \\
(A \otimes (B \otimes C)) \pitchfork D \arrow[u, "a_{ABC} \pitchfork 1"] \arrow[rr, "p_{A(B \otimes C)D}"'] & & A \pitchfork ((B \otimes C) \pitchfork D) \arrow[u, "1_A \pitchfork p_{BCD}"']
\end{tikzcd}$$

MCC3 The following diagram commutes:

$$\begin{tikzcd}
C \pitchfork D \arrow[r, "L^{A \otimes B}_{CD}"] \arrow[d, "L^B_{CD}"'] & ((A \otimes B) \pitchfork C) \pitchfork ((A \otimes B) \pitchfork D) \arrow[dd, "1 \pitchfork p"] \\
(B \pitchfork C) \pitchfork (B \pitchfork D) \arrow[d, "L^A_{B \pitchfork C, B \pitchfork D}"'] & \\
(A \pitchfork (B \pitchfork C)) \pitchfork (A \pitchfork (B \pitchfork D)) \arrow[r, "p \pitchfork 1"'] & ((A \otimes B) \pitchfork C) \pitchfork (A \pitchfork (B \pitchfork D))
\end{tikzcd}$$

MCC4 The following diagram commutes:

$$\begin{tikzcd}
(A \otimes K) \pitchfork B \arrow[rr, "p_{AKB}"] & & A \pitchfork (K \pitchfork B) \\
& A \pitchfork B \arrow[ul, "r_A \pitchfork 1_B"] \arrow[ur, "1_A \pitchfork i_B"'] &
\end{tikzcd}$$

Definition 5.3.2 [3,p. 535] A symmetric monoidal closed category is a monoidal closed category with a symmetry (as defined in Definition 5.1.2).

The category C, with $\otimes$ = the categorical product bifunctor, and K = terminal object, is easily expandable into a monoidal category. If this category is expandable into a monoidal closed category then C is said to be cartesian closed. This last concept can be defined in the following convenient way.

Definition 5.3.3 [3,p. 550], [4,pp. 2-3] A cartesian closed category consists of a category C such that:

(i) there exists finite products $A \times B$, $A, B \in C$;

(ii) $- \times B$ has a right adjoint for all $B \in C$;

(iii) there exists a terminal object in C.

Definition 5.3.4 [9,p. 127] A biclosed category is a monoidal category for which the endofunctors $A \otimes -$ and $- \otimes B$ both have right adjoints:

$$\hom(A \otimes B, C) \cong \hom(A, B \pitchfork C) \cong \hom(B, A \,\tilde{\pitchfork}\, C)$$

where $B \pitchfork -$ is the right adjoint of $- \otimes B$ and $A \,\tilde{\pitchfork}\, -$ is the right adjoint of $A \otimes -$, for all objects A and B.

These are the biautonomous categories recommended by MacLane in [9, p. 127].

§5.4 Relations Between the Data and Relations Between the Axioms of a Monoidal Closed Category

Throughout this section we assume that:

(i) C is a category ;

(ii) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor ;

(iii) $\pitchfork : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor ;

(iv) ι ; $\hom(A \otimes B, C) \to \hom(A, B \pitchfork C)$ is a natural equivalence.

We now give a list of the relations between the data and relations between the axioms for structured categories, given by Eilenberg and Kelly. [3, pp. 477-482] shows that there exist bijections between the following pairs of data;

(1) the set of natural equivalences $\ell = \ell_A : K \otimes A \to A$ and the set of natural equivalences $\nu = \nu_{AB}$; $\hom(A,B) \to \hom(K, A \pitchfork B)$;

(2) the set of natural transformations ν and the set of natural transformations $j = j_A : K \to A \pitchfork A$;

(3) the set of natural equivalences $r = r_A : A \otimes K \to A$ and the set of natural equivalences $i = i_A : A \to K \pitchfork A$.

(4) the set of natural equivalences $a = a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ and the set of natural equivalences $p = p_{BCD} : (B \otimes C) \pitchfork D \to B \pitchfork (C \pitchfork D)$;

(5) the set of natural transformations $p = p'_{ABC} : (A \otimes B) \pitchfork C \to A \pitchfork (B \pitchfork C)$ and the set of natural transformations $L^B_{AC} : A \pitchfork C \to (B \pitchfork A) \pitchfork (B \pitchfork C)$.

Eilenberg and Kelly have also shown that the following relations exist between the axioms:

Suppose that in our assumed basic situation we have natural transformations j and L and natural equivalences a, ℓ, r, p, ν, i then the following implications hold between the axioms MC, MCC, and CC.

(i) In the presence of CC6 we have MC1 <=> ι <=> CC2.

(ii) MC2 <=> MCC1 <=> CC3.

(iii) MC3 <=> MCC2 <=> MCC3 <=> CC4.

(iv) MC4 <=> MCC4 <=> CC5.

(v) CC6 => MC5.

(vi) MC2 is a consequence of MC1, MC3, MC4, MC5.

This is Proposition 4.1 of [3, p. 482] and Proposition 4.2 of [3, p. 486].

Theorem 5.4.1 A monoidal category for which $- \otimes B$ has a right adjoint for all B, is monoidal closed.

Proof: This follows immediately from the definitions involved, the results of this section, and [8, p. 100, Theorem 3].

Theorem 5.4.2 A closed category for which

(i) $B \textcircled{h} -$ has a left adjoint $- \otimes B$ for all B;

and (ii) there exists a natural equivalence

$a = a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for all A, B, C,;

is monoidal closed.

Proof: This follows from the results and definitions of this section and [8, p. 100, Theorem 3].

§5.5 Posets and Categories

We shall use the notation $Cat(P, \leq)$ for the category associated with the poset $(P, \leq)$. The objects of $Cat(P, \leq)$ are the elements of $(P, \leq)$, and the morphisms of $Cat(P, \leq)$ are the valid $\leq$ of $(P, \leq)$. Many of our concepts related to posets are particular cases of more general ideas concerning categories. In this section we seek to clarify these generalizations, and conclude with a remark concerning the apparent "symmetry" between the data for monoidal and closed categories.

POSET CONCEPT	CATEGORICAL CONCEPT
1. Poset $(P,\leq)$ [Definition 1.1.1]	1'. $Cat(P,\leq)$ will denote the category associated with the poset $(P,\leq)$. [7, p. 508].
2. The dual of a poset $(P,\leq)$ i.e. $(P,\geq)$ [Definition 1.1.6].	2'. The dual of a category C, i.e. C^{op} [8, p. 33].
3. $f : (P,\leq) \to (Q,\leq)$ is a morphism of posets. [Definition 1.1.14].	3'. $f : Cat(P,\leq) \to Cat(Q,\leq)$ is a functor. [7, p. 51?].
4. $f : (P,\leq) \to (Q,\leq)$ and $g : (Q,\leq) \to (P,\leq)$ are morphisms of posets and f is a left adjoint to g (and g is a right adjoint to f). [Definition 2.1.1].	4'. $f : Cat(P,\leq) \to Cat(Q,\leq)$ and $g : Cat(Q,\leq) \to Cat(P,\leq)$ are functors and f is a left adjoint to g (g is a right adjoint to f). [7, p. 532].
5. Theorem 2.1.1, page 20 Theorem 2.1.2, page 20 (dual).	5'. [7, p. 532, Theorem 6 and 7, p. 533 Corollary].
6. First element of a poset. [Definition 1.1.4].	6'. Initial object of a category. [7, p. 512].
7. Last element of a poset. [Definition 1.1.4].	7'. Terminal object of a category. [7, p. 512].

8. The meet $a \wedge b$ of two elements a, b in a poset. [Definition 1.1.8].	8'. The product $A \times B$ of two objects A, B in a category. [7, p. 528].
9. Proposition 2.1.3 (part 1) and Proposition 2.1.4 (part 1) (dual), page 21.	9'. Any functor with a left adjoint (dually right adjoint) preserves terminal objects (dually initial objects).
10. A bimorphism $\theta : (P,\leq) \times (Q,\leq) \to (R,\leq)$ [Definition 1.1.16].	10'. A bifunctor $\beta : \mathrm{Cat}(P,\leq) \times \mathrm{Cat}(Q,\leq) \to \mathrm{Cat}(R,\leq)$ [7, p. 515].
11. Theorem 2.2.1, page 21 Theorem 2.2.2, page 21 (dual).	11'. [8, p. 100, Theorem 3].

The structured categories of Chapter V involve axioms concerning commutative diagrams of morphisms in the underlying category; if this category is of the form $\mathrm{Cat}(P,\leq)$ then it follows from the Lemma below that these conditions are void.

Lemma 5.5.1 If $(P,\leq)$ is a poset, then every diagram in $\mathrm{Cat}(P,\leq)$ commutes.

Proof: Any two routes around a diagram in $\mathrm{Cat}(P,\leq)$, say a to b, represent the single existing morphism $a \to b$.

The proofs of some of the results below are obvious from the definitions involved and are not given.

Proposition 5.5.2 There is a 1 - 1 correspondence between

(a) the morphisms : $(P,\leq) \to (Q,\leq)$

and (b) the functors : $\mathrm{Cat}(P,\leq) \to \mathrm{Cat}(Q,\leq)$.

Proposition 5.5.3 There is a 1 - 1 correspondence between

(a) the bimorphisms $\theta : (P,\leq) \times (Q,\leq) \to (R,\leq)$

and (b) the bifunctors $\beta : \mathrm{Cat}(P,\leq) \times \mathrm{Cat}(Q,\leq) \to \mathrm{Cat}(R,\leq)$.

Proposition 5.5.4 Let $f : (P,\leq) \to (Q,\leq)$ and $g : (Q,\leq) \to (P,\leq)$ be morphisms of posets. Then f is a left adjoint to g if and only if the associated functor $\underline{f} : \mathrm{Cat}(P,\leq) \to \mathrm{Cat}(Q,\leq)$ is a left adjoint in the categorical sense, to the associated functor $\underline{g} : \mathrm{Cat}(Q,\leq) \to \mathrm{Cat}(P,\leq)$.

Proposition 5.5.5 Given bimorphisms $\otimes : (P,\leq) \times (Q,\leq) \to (R,\leq)$ and $\pitchfork : (Q,\geq) \times (R,\leq) \to (P,\leq)$, then $\otimes$ is a left adjoint to $\pitchfork$ if and only if the associated bifunctor $\otimes : \mathrm{Cat}(P,\leq) \times \mathrm{Cat}(Q,\leq) \to \mathrm{Cat}(R,\leq)$ is a left adjoint to the associated bifunctor $\pitchfork : \mathrm{Cat}(Q,\geq) \times \mathrm{Cat}(R,\leq) \to \mathrm{Cat}(P,\leq)$.

Theorem 5.5.6 Let $(P,\leq)$ be a poset. There is a 1 - 1 correspondence between

(a) the set of monoidal poset structures on $(P,\leq)$

and (b) the set of monoidal category structures on $\mathrm{Cat}(P,\leq)$.

Proof: This result of [3, p. 554] follows from Lemma 5.5.1 and Proposition 5.5.3.

Corollary 5.5.7 There is a 1 - 1 correspondence between

(a) the set of symmetric monoidal poset structures on $(P,\leq)$

and (b) the set of symmetric monoidal category structures on $\mathrm{Cat}(P,\leq)$.

Proof: This is immediate from Theorem 5.5.6.

Theorem 5.5.8 Let $(P,\leq)$ be a poset. There is a 1 - 1 correspondence between

(a) the set of closed poset structures on $(P,\leq)$

and (b) the set of closed category structures on $Cat(P,\leq)$.

Proof: Suppose $(Cat(P,\leq),\pitchfork,K,i,j,L^A_{BC})$ is a closed category structure. The bifunctor $\pitchfork : Cat(P,\geq) \times Cat(P,\leq) \to Cat(P,\leq)$ corresponds to the bimorphism $\pitchfork : (P,\geq) \times (P,\leq) \to (P,\leq)$ by Proposition 5.5.3. The object K in $Cat(P,\leq)$ is the element k in $(P,\leq)$. The existence of the natural equivalence $i = i_A : A \to K \pitchfork A$ and the natural transformation $L = L^A_{BC} : B \pitchfork C \to (A \pitchfork B) \pitchfork (A \pitchfork C)$ correspond to $a = k \pitchfork a$ and $b \pitchfork c \leq (a \pitchfork b) \pitchfork (a \pitchfork c)$ respectively in the closed poset structure. The natural equivalence $hom(A,B) \approx hom(K,A \pitchfork B)$ of axiom CC1 implies that $a \leq b \iff k \leq a \pitchfork b$ in the closed poset structure. Notice that this last bijection implies $k \leq a \pitchfork a$ (in the closed poset) which corresponds to the existence of the natural transformation $j = j_A : K \to A \pitchfork A$ in the closed category structure.

Conversely, let us assume that $(P,\leq,\pitchfork,k)$ is a closed poset structure with an associated category $Cat(P,\leq)$. Since the objects of $Cat(P,\leq)$ are the elements of $(P,\leq)$, the element k in $(P,\leq)$ is the object K in $Cat(P,\leq)$. The bimorphism $\pitchfork : (P,\geq) \times (P,\leq) \to (P,\leq)$ corresponds to the bifunctor $\pitchfork : Cat(P,\geq) \times Cat(P,\leq) \to Cat(P,\leq)$ by Proposition 5.5.3. Since the morphisms in $Cat(P,\leq)$ are the valid $\leq$ of $(P,\leq)$, the remaining data (a), (b), (c) in a closed poset structure (see Definition 3.1.1) correspond to the existence of the natural equivalence $i = i_A : A \to K \pitchfork A$, the natural transformation $L = L^A_{BC} : B \pitchfork C \to (A \pitchfork B) \pitchfork (A \pitchfork C)$ and the natural equivalence $\hom(A,B) \approx \hom(K, A \pitchfork B)$ of axiom CC1 respectively. We notice (c) of the data of a closed poset implies $k \leq a \pitchfork a$ and this corresponds to the existence of the natural transformation $j = j_A : K \to A \pitchfork A$. The axioms CC2-CC6 in a closed category structure are satisfied by Lemma 5.5.1. Hence the result follows.

Theorem 5.5.9 [3, p. 555] There is a 1 - 1 correspondence between

(a) the set of monoidal closed poset structures on $(P, \leq)$

and (b) the set of monoidal closed category structures on $\mathrm{Cat}(P, \leq)$.

Proof: This follows from Theorems 5.5.6 and 5.5.8.

Corollary 5.5.10 There is a 1 - 1 correspondence between

(a) the set of symmetric monoidal closed poset structures on $(P, \leq)$

and (b) the set of symmetric monoidal closed category structures on $\mathrm{Cat}(P, \leq)$.

Proof: This is immediate.

Corollary 5.5.11 $(P, \leq)$ has a cartesian closed poset structure if and only if $\mathrm{Cat}(P, \leq)$ has a cartesian closed category structure.

Proof: This follows easily from Theorem 5.5.9, and 7 - 7', 8 - 8' in the above table.

Corollary 5.5.12 There is a 1 - 1 correspondence between

(a) the set of biclosed poset structures on $(P, \leq)$

and (b) the set of biclosed category structures on $\mathrm{Cat}(P, \leq)$.

Proof: This follows from Theorem 5.5.9.

Remark: It is now clear (of course) that our results in §3.2 and Theorems 3.3.1 and 3.3.3 are particular cases of results of [3], namely the list on page 56 and Theorems 5.4.1 and 5.4.2 respectively.

Remark: Consider the basic situation described at the beginning of §5.4 pages 55-56 (data (i) ... (iv)).

If K is an object of C and the following additional data are satisfied

(a) there exists a natural equivalence $i = i_A : A \to K \pitchfork A$ for all $A \in C$;

(b) there exists a natural transformation $j = j_A : K \to A \pitchfork A$ for all $A \in C$;

(c) there exists a natural transformation

$L = L^A_{BC} : B \pitchfork C \to (A \pitchfork B) \pitchfork (A \pitchfork C)$ for all $A, B, C \in C$;

plus axioms CC1 - CC6 then $(C, \pitchfork, K, i, j, L)$ is a closed category.

If K is an object of C and the following data are satisfied

(a') there exists a natural equivalence $r = r_A : A \otimes K \to A$ for all $A \in C$;

(b') there exists a natural equivalence $\ell = \ell_A : K \otimes A \to A$ for all $A \in C$;

(c') there exists a natural equivalence

$a = a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for all $A, B, C \in C$;

plus axioms MC1 - MC5 then $(C, \otimes, K, r, \ell, a)$ is a monoidal category.

There is seemingly a nice symmetry amongst this data since

(a) <=> (a') by (3) page 56

(b) <=> (b') by (1), (2) page 56

also by (4) and (5) page 56 (c') => (c). This symmetry would be complete if it could be shown that (c) => (c'), and it is by no means obvious if this is so. However, it follows from our seven examples of non-monoidal closed posets, pages 42-3 that there does not exist a bijection between the set of natural transformations $L^B_{AC} : A \pitchfork C \to (B \pitchfork A) \pitchfork (B \pitchfork C)$ and the set of natural equivalences $a = a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, for otherwise these seven examples would be monoidal closed posets. Hence we have shown that although (c') => (c), the converse is false. Therefore the symmetry between the definition of monoidal category and the definition of closed category is not complete.

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